Nov. 10 / 2017

Coming Wed. Nov. 15 midterm 2.
(Covers material after midterm 1).
↳ para. curves + polar coor. + seq. & series.

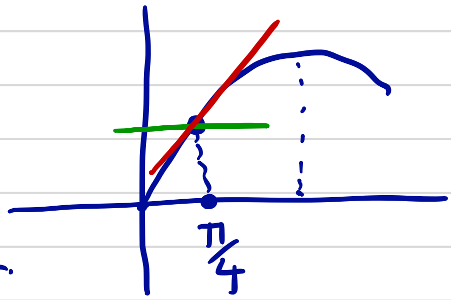
- Taylor series/polynomial is on the test.

- $f(x)$ given function ⇝ f is many times differentiable.

  $\underline{\underline{a}}$ point in the domain of f

$(f(x), a)$ ⇝ power series (Taylor series) (representation of f around $\underline{\underline{a}}$).

Approximating f by polynomials (of higher & higher degree) near a point $\underline{\underline{a}}$.



Ex. $f(x) = \sin(x)$ $\quad a = \pi/4$

⓪ deg 0 poly. Const. function

$y = \sin(\pi/4) = \sqrt{2}/2$. ⇝ best Const. approx.

① deg 1 poly. linear (best) function approx.

Recall eqn. of tangent line: $\dfrac{y - f(a)}{x - a} = f'(a)$.

$y = f(a) + f'(a)(x-a)$

$y = \sin(\pi/4) + \sin'(\pi/4)(x - \pi/4) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}(x - \pi/4)$.

deg 2

. The best quad. approx. of f at point a is :

$$y = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

. The best deg. n approx. of f at a :
(deg n Taylor poly. of f at a)

$$y = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = T_n(x)$$

(For this you need $f(a), f'(a), \ldots, f^{(n)}(a)$).

Theorem This polynomial (i.e. Taylor poly.) is the "best" approx. of f around a by a deg. n polynomial.

. $T_n(a) = f(a)$ $\quad T_n'(a) = f'(a)$

$T_n''(a) = f''(a)$ ---- $T_n^{(n)}(a) = f^{(n)}(a)$.

Ex. $n=2$ $\quad T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

f & a given

$T_2(a) = f(a)$

$T_2'(x) = f'(a) + \frac{f''(a)}{\cancel{2}} \cdot \cancel{2}(x-a) \rightsquigarrow T_2'(a) = f'(a)$.

$T_2''(x) = f''(a) \rightsquigarrow T_2''(a) = f''(a)$.

Ex. $f(x) = \sin(x)$ & $a = \pi/4$

$\sin(\pi/4) = \frac{\sqrt{2}}{2}$ $\quad \sin'(\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$ $\quad \sin''(\pi/4) = -\sin(\pi/4) = -\frac{\sqrt{2}}{2}$ ...

$$T_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x-\tfrac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\tfrac{\pi}{4})^2$$

Taylor series (of f at $x=a$):

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

- Many times taking $a=0$ is more convenient. (Maclaurin Series)

- There are these possibilities
  1. $T(x)=f(x)$ for all x
  2. - - - for x near a ($|x-a| < R$) radius of conv.
  3. $T(x)=f(x)$ only at $x=a$

- Examples we deal with are 1 & 2

Ex. $f(x) = e^x$, $a=0$.

$f^{(n)}(x) = e^x$ for any $n>0$.

$f^{(n)}(0) = e^0 = 1$.

Taylor series of e^x at $a=0$ $= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

One shows $e^x = T(x)$ for all x.

About error of Taylor poly. approx.

. $T_n(x)$ = deg n Taylor poly. of f at $x=a$.

$$\underbrace{R_n(x) = f(x) - T_n(x)}_{\text{error of approx.}}$$

Theorem $$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

for some z between x & a.

Using this one can show in many cases that $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$.

Rem This theorem follows from Mean Value Th.

Ex. $f(x) = e^x$ and $a=0$. $\quad 0<z<x$

Fix some x.

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} = \frac{e^z}{(n+1)!} x^{n+1} \leq \frac{e^x \cdot x^{n+1}}{(n+1)!}$$

x fixed, $n \longrightarrow \infty$ then $\dfrac{e^x x^{n+1}}{(n+1)!} \longrightarrow 0$ So $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$.

$$\lim_{n\to\infty} R_n(x) = 0.$$

Ex. Taylor series of $f(x) = \sin(x)$, $a = 0$.

$\sin(x)$ $\quad$ $\sin'(x) = \cos(x)$ $\quad$ $\sin''(x) = -\sin(x)$ $\quad$ $\sin'''(x) = -\cos(x)$

$\sin^{(4)}(x) = \sin(x)$

$$\sin(x) = T(x) = 0 + x + \frac{0}{2!}x^2 + (-1)\frac{x^3}{3!} + 0\frac{x^4}{4!}$$

(Shown using error estimate theorem)

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$