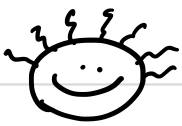
Nov. 1 / 2017



8.3 Integral & Comparison tests.

- $a_1, a_2, \ldots \rightsquigarrow$ series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$

Goal Given the a_n decide if $\sum_{n=1}^{\infty} a_n$ converges or diverges.

- We will see a few different "tests" for conv/div. of series.

Integral test

- $a_n = f(n)$ f some (contin.) function
- f decreasing & positive function (a_n is dec. seq.).

(Recall for $\sum_{n=1}^{\infty} a_n$ to conv. it is necessary that $\lim_{n\to\infty} a_n = 0$.)

So $\lim_{x\to+\infty} f(x) = 0$.

(but $\int_1^{\infty} f(x)\,dx \neq \sum_{n=1}^{\infty} f(n)$)

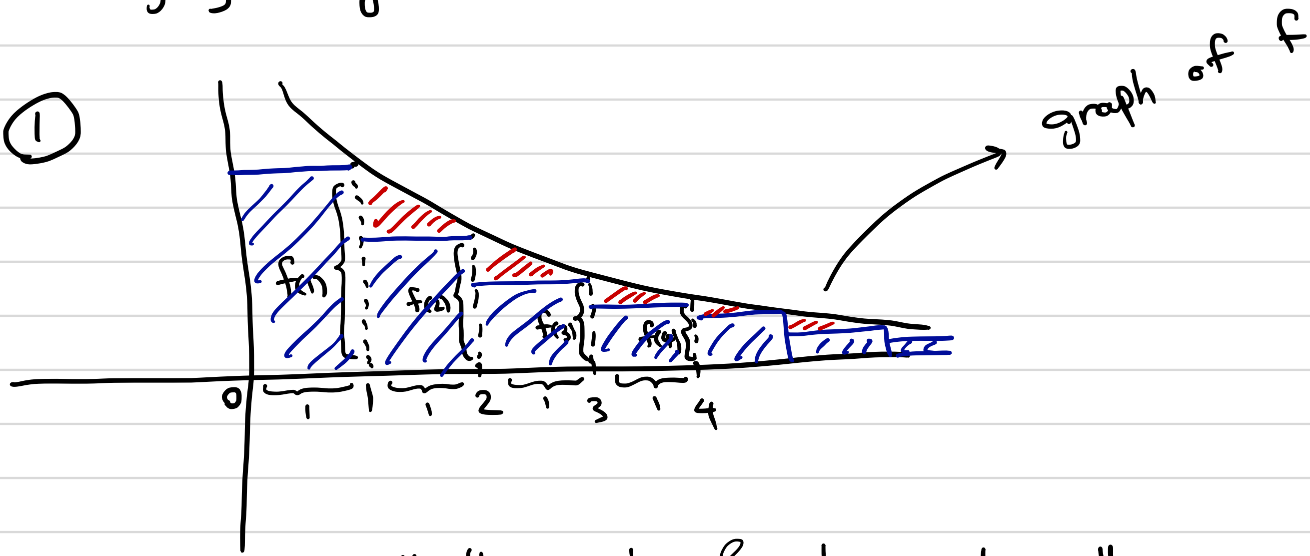
Theorem / General fact

① If $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

② If $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ div.

Justifying why this is true:

1



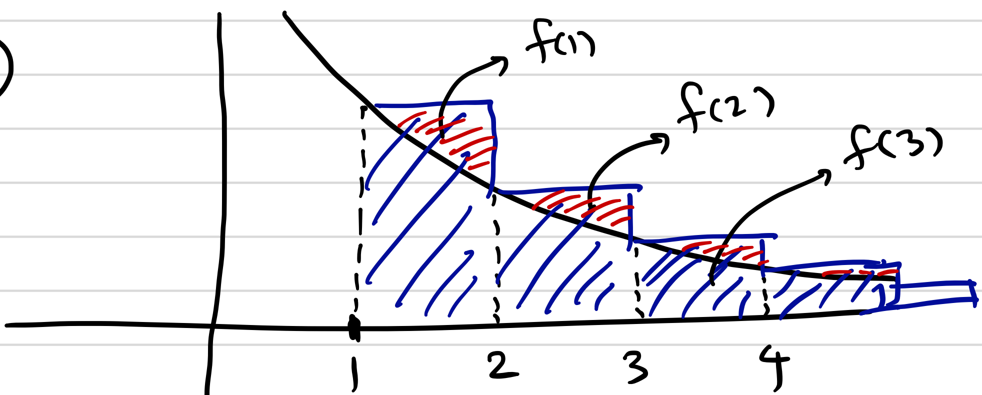
Observation: All the rectangles lie under the graph of f

- Sum of areas of the rect. is exactly $\sum_{n=1}^{\infty} f(n)$
- Sum of areas of rect. (starting from $n=2$) $\leq$ area under f from $x=1$ to ∞

$f(1) + f(2) + f(3) \cdots$ conv. $\iff$ $f(2) + f(3) + \cdots$ conv.

(we can drop the first term, i.e. $f(1)$).

2



- Sum of areas of rect. $\geq$ area under the graph of f from $x=1$ to ∞.

• <u>Rem</u> $\int_1^\infty f(x)\,dx$ is usually easier to ~~find~~ Compute than the series $\sum_{n=1}^{\infty} f(n)$.

<u>Example</u> $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \cdots$ Harmonic series

Using int. test we see this is div.

$$\int_1^\infty \frac{1}{x}\,dx \rightsquigarrow \ln(x)\Big|_1^\infty = \underbrace{\ln(\infty)}_{\lim_{x\to\infty}\ln(x)} - \underbrace{\ln(1)}_{0}$$

$$= \infty .$$

<u>Example</u> $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

We show this is Conv.

$$\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = \underbrace{\cancel{\left(-\frac{1}{\infty}\right)}^{0}}_{\lim_{x\to\infty}\frac{-1}{x}} - \left(-\frac{1}{1}\right) = 1$$

But, as I mentioned, $\sum_{n=1}^{\infty} \frac{1}{n^2} \neq 1$ in fact equal to $\frac{\pi^2}{6}$.

Recall $0 < P$ fixed

- $\int_1^\infty \frac{1}{x^P}\,dx$ Convergent if $1 < P$
- $\int_1^\infty \frac{1}{x^P}\,dx$ divergent if $0 < P \leq 1$

- $\sum_{n=1}^{\infty} \frac{1}{n^P}$ Conv. $1 < P$
- $\sum_{n=1}^{\infty} \frac{1}{n^P}$ div. $0 < P \leq 1$

<u>Ex.</u> $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$ div.

$(P = \frac{1}{2})$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ "

<u>Rem</u> Geometric series $\sum_{n=0}^{\infty} r^n \; \left(= \frac{1}{1-r}\right)$

(Sum of powers of a fixed r)

base is r fixed, power or exponent is n variable.

$\sum_{n=1}^{\infty} \frac{1}{n^P}$ power is p fixed

base is n variable.

Comparison test

- If bigger series Conv. $\Rightarrow$ small series Conv.
- If smaller div. $\Rightarrow$ bigger is div.

Ex. $\sum_{n=1}^{\infty} \frac{1}{2^n + \ln(n)}$ $\boxed{\text{Conv.}}$ or div.?

$$\frac{1}{2^n+\ln(n)} \leq \frac{1}{2^n} \qquad (2^n+\ln n \geq 2^n)$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n+\ln n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \rightsquigarrow \text{Conv.}$$

(geo. series with $r=\frac{1}{2}$)

Theorem (Comparison test)

$a_n, b_n \geq 0$

- $a_n \leq b_n$ for all n $\quad \sum_{n=1}^{\infty} b_n$ Conv. $\Rightarrow \sum_{n=1}^{\infty} a_n$ Conv.
- $a_n \geq b_n$ - - - - $\quad$ - - - - div. $\Rightarrow$ - - - div.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Conv. or div. ?

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-2}$ Conv. or div. ?

(we compute the answer using partial fraction)

$$\frac{1}{n^2-2} < \frac{1}{(n-1)^2}$$

(if $n \geq 2$)

$(n-1)^2 = n^2 - 2n + 1$

$n^2 - 2n + 1 < n^2 - 2$

(if $n \geq 2$)

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{Conv.}$$

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-2}$ also Conv. $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-2}$ also Conv.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+1}$ Conv. div ?

$$\frac{1}{3n^2+1} < \frac{1}{3n^2}$$
