Dec. 8 / 2017

- Monday $\leadsto$ extra office hour 1–3
- Try working on the sample finals. (posted) $\to$ Sample final 1 has answers.
- Review midterms

Sample final 3

#9 $y' = y + x \leadsto$ not separable

$y(0) = 2$

- Separable eqn.
- multi. by int. factor

Use "multi. by int. factor".

$y' + P(x)\,y = Q(x)$

$$y' - y = x$$

$$\overset{?}{I(x)}\,(y' - y) = I(x)\,x$$

we know $I(x) = A\,e^{\int P(x)dx}$

$P(x) = -1 \leadsto \int -1 = -x$

e^{-x}

$$\underbrace{I(x)y' - I(x)y}_{(I(x)y)'}$$

$$\underbrace{e^{-x}y' - e^{-x}y}_{(e^{-x}y)'} = e^{-x}\cdot x$$

$$(e^{-x}y)' = e^{-x}\cdot x \leadsto e^{-x}y = \int x e^{-x}\,dx + C$$

integration by parts

$$\left(\int \underset{u}{x}\,\underset{dv}{e^{-x}dx} = uv - \int v\,du = -xe^{-x} + \int e^{-x} = -xe^{-x} - e^{-x}\right)$$

$du = 1 \quad v = -e^{-x}$

$$e^{-x} y = -x e^{-x} - e^{-x} + C$$

$$y(x) = -x - 1 + \frac{C}{e^{-x}} = -x - 1 + Ce^{x}$$

$$\left(y' = -1 + Ce^{x} = y + x = \cancel{-x} - 1 + Ce^{x} + \cancel{x} \right).$$

☺

Find const. C using initial condition $y(0) = 2$.

$$y(0) = -0 - 1 + C\underbrace{\cancel{e^{0}}}_{1} = C - 1 = 2 \Rightarrow C = 3.$$

$$y(x) = -x - 1 + 3e^{x}$$ ☺

Review of Taylor series

Sample final 3

\# 14 Taylor series at $x = 0$ $(a = 0)$.

- $$\frac{1}{1+3x} = \frac{1}{1-(-3x)} = \sum_{n=0}^{\infty} (-3)^n x^n.$$

$$\left(\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{Conv. for } |x| < 1 \right)$$

$$\underbrace{|-3x|}_{3|x|} < 1 \rightsquigarrow |x| < \frac{1}{3}.$$

- $$\frac{1}{(1+3x)^2} = \left(\frac{-1}{3}\right)\left(\frac{1}{1+3x}\right)' = \left(\frac{-1}{3}\right)\left(\sum_{n=0}^{\infty} (-3)^n x^n\right)'$$

$$= \left(-\frac{1}{3}\right)\left(\sum_{n=1}^{\infty} (-3)^n n\, x^{n-1}\right) = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)(-3)^n n\, x^{n-1}.$$

- $\sqrt{1+3x}$

$\sqrt{1+x}$ or more generally $(1+x)^k$

k any number.

- $k=2$ $\quad (1+x)^2 = 1+2x+x^2$

$k=3$ $\quad (1+x)^3 = 1+3x+3x^2+x^3$

k natural number $=1,2,3,\ldots$ $\quad (1+x)^k = 1 + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{k-1}x^{k-1} + x^k$

$$\binom{k}{n} = \frac{k \times (k-1) \times (k-2) \times \cdots \times (k-n+1)}{\underbrace{n \times (n-1) \times \cdots \, 1}_{n!}}$$

General binomial theorem (series form)

works for any k

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^k$$

$$= 1 + \frac{k}{1}x + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$\sqrt{1+x} = 1 + (1/2)x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2 + \cdots$$

$k=\frac{1}{2}$

Another example of binomial series:

$k=-1$

$$(1+x)^{-1} = 1 + \frac{(\cancel{-1})^{-1}}{\cancel{1}}x + \frac{\cancel{(-1)(-2)}^{1}}{\cancel{2!}}x^2 + \cdots$$

$$\hookrightarrow \frac{1}{1+x} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$
$$= 1 - x + x^2 - x^3 + \cdots$$

- Also you can use binomial series to get series for $(1+x)^{-2}$.
  $k=-2$

---

15 (Taylor poly. or first few terms of Taylor series)

Write first four terms in Taylor series of $f(x)=\sqrt{x}$ at $x=4$.

Use Taylor formula:

$$f(4) + f'(4)(x-4) + \frac{f''(4)(x-4)^2}{2!} + \frac{f'''(4)(x-4)^3}{3!}$$

$$x^{\frac{1}{2}} \xrightarrow{\ '\ } \frac{1}{2}x^{-\frac{1}{2}} \left(\frac{1}{2\sqrt{x}}\right) \xrightarrow{\ '\ } -\frac{1}{4}x^{-\frac{3}{2}} \longrightarrow \frac{3}{8}x^{-\frac{5}{2}}$$

$x=4$: 2, $\frac{1}{4}$, $\frac{1}{4}\cdot\frac{1}{8}$, $\frac{3}{8}\cdot\frac{1}{32}$