April 8 Zoom

Recall $X = Bl_0(\mathbb{A}^n) = \{ (v, [\ell]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid v \in \ell \}$

$$\begin{tikzcd} & X \arrow[dl, "\pi = pr_1"', "\text{Blow up}"] \arrow[dr, "p = pr_2", "\text{line bundle}"'] & \\ \mathbb{A}^n & & \mathbb{P}^{n-1} \end{tikzcd}$$

$p : X \longrightarrow \mathbb{P}^{n-1}$ is the most important ex. of a "line bundle" called "tautological bundle" on $\mathbb{P}^{n-1}$.

Let Z be a variety & $p : L \longrightarrow Z$ a morphism of varieties s.t.

1. $\forall z \in Z$, $p^{-1}(z) \cong \mathbb{A}^1$ fibers are lines

$\Uparrow$

2. $\forall z \in Z$, $\exists U \subset Z$, $z \in U$ & φ iso. between $p^{-1}(U)$ & $U \times \mathbb{A}^1$ that commutes with

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi", "\cong"'] \arrow[d, "p"'] & U \times \mathbb{A}^1 \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

3. Each $p^{-1}(z)$ is a 1-dim v.s. & φ is lin. isoism

(same def./notion for top. spaces/manifolds etc.)

Fiber bundle $\mathbb{A}^1 \longleftrightarrow F$ some variety)

vec. " $\mathbb{A}^1 \longleftrightarrow \mathbb{A}^r$

Ex. X smooth curve

TX = tangent bundle of X

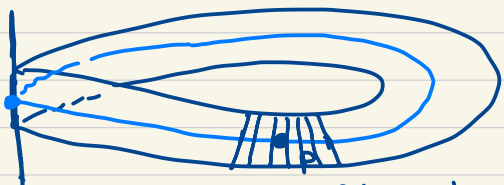
$$= \bigcup_{p \in X} T_p X$$

tangent space to X at point p

Ex. $k = \mathbb{R}$ $\quad X =$ circle

$L =$ (infinite) Möbius strip

$L \not\cong X \times \mathbb{R} \longrightarrow$ cylinder



Locally trivial

Exercise: Check $X = Bl_0(\mathbb{A}^n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$

$$\downarrow p = pr_2$$

$$\mathbb{P}^{n-1}$$

is a line bundle over $\mathbb{P}^{n-1}$!

Rem There is an operation of tensor product of line bundles.

$$\begin{array}{c} L_1 \\ p_1 \downarrow \\ X \end{array} \quad \& \quad \begin{array}{c} L_2 \\ p_2 \downarrow \\ X \end{array} \quad \rightsquigarrow \quad \begin{array}{c} L_1 \otimes L_2 \\ p \downarrow \\ X \end{array}$$

- Fact: The set of line bundles over X form a group (!) called $\mathrm{Pic}(X)$. Picard group of X. Trivial line bundle $X \times \mathbb{A}^1$ is the identity element.

Back to blow-up:

(See Eisenbud "Comm. alg. with a view towards alg. geo." p. 148 Sec. 5.2 Blow up alg.)

X affine alg. var. $Y \subset X$ subvar.

Z = Blow up of X along Y.

$a_1, \ldots, a_r$ k-alg. gen. for $\underline{k[X]}$

$g_0, \ldots, g_s$ gen. of ideal $\underline{I = I(Y)} \subset k[X]$

$R = k[X]$ $\qquad$ aux. var.

$$k[x_1, \dots, x_r, y_0, \dots, y_s] \xrightarrow{\varphi} R[t] = \bigoplus_{i \geq 0} R t^i$$

$$x_i \longmapsto a_i$$

$$y_j \longmapsto g_j t$$

Exercise: Image of φ is the alg.: $\underbrace{R \oplus It \oplus I^2 t^2 \oplus \cdots}_{\text{Blow-up alg. of } I}$ $\quad$ ($R = I^0$)

- kernel of φ is homog. in the y_j.

& blow up of X along Y to be subvar. of $\mathbb{A}^r \times \mathbb{P}^s$ defined by this ideal.

Ex. Blow-up alg. of $m = m_0 \subset \overbrace{k[x_1, \dots, x_n]}^{R}$

$$R \oplus m \oplus m^2 \oplus \cdots$$

$$\begin{tikzcd} Z = Bl_Y(X) \arrow[d, "\pi"] \arrow[r, phantom, "\subset"] & \mathbb{A}^r \times \mathbb{P}^{s-1} \arrow[d, "\pi"'] \\ X \arrow[r, phantom, "\subset"] & \mathbb{A}^r \end{tikzcd}$$

$$k[Z] \cong R \oplus I \oplus I^2 \oplus \cdots$$

hybrid affine + homog. coor.
quotient of $k[x_1 \cdots x_r, y_0 \cdots y_s]$

Exercise $\quad k[\pi^{-1}(Y)] \cong R/I \oplus I/I^2 \oplus \cdots$

<u>Def.</u> $I \subset R$

$R \oplus I \oplus I^2 \oplus \cdots \subset R[t] \longrightarrow$ Blowup alg. or Rees alg.

$R/I \oplus I/I^2 \oplus \cdots \longrightarrow$ associated graded alg. of (R, I)

($\pi^{-1}(Y)$ related to normal cone of Y in X)

Aside:

Algebraically you can generalize the construction to a decreasing sequence of ideals.

$$I_\bullet = (I_0 \supset I_1 \supset I_2 \supset \cdots) \quad \text{ideals in } R$$

$$\mathcal{R}(I_\bullet) = \bigoplus_{i \geq 0} I_i t^i$$

$$\operatorname{gr}_{I_\bullet} R = \bigoplus_{i \geq 0} I^i / I^{i+1} .$$

Geometric meaning of $Bl_Y(X)$:

At $p \in Y$ you look at <u>normal cone</u> of Y in X at p, projectivize it & attach it to p.

<u>Rem</u> Blow up is used in resolution of singularities.

Hironaka: res. of sing. can be achieved by a sequence of blow-ups.

Changing gears:

Introduction to "intersection theory"

Problem
$f_1, \dots, f_n$ given poly. in $k[x_1, \dots, x_n]$

- Find # of sol. of $f_1 = \dots = f_n = 0$.
- Find, the solution of the system
  Compute
  - f_i's linear $\rightsquigarrow$ Gaussian elimination
  - f_i's poly. $\rightsquigarrow$ well-known algo.
    & k alg. closed in Gröbner basis theory
    (see Cox-Little-O'Shea "Using alg. geo.")

(very general)
Problem (of intersec. theory)

X variety

$Y_1, \dots, Y_r \subset X$ subvar.

Describe $Y_1, \dots, Y_r$.

__Examples__ (of results in intersec. theory)

Bezout theorem: $f(x,y,z)$ homog. poly.
$g(x,y,z)$
of deg. d_1 & d_2.

$$X = V(f) \ \& \ Y = V(g) \subset \mathbb{P}^2$$

$$\underbrace{|X \cap Y|}_{\text{counted with multiplicity each intersec. point}} = d_1 d_2$$

__Def.__ (+ theorem)

$X \subset \mathbb{P}^N$ proj. subvariety of dim d

$$\deg(X) := |X \cap L|$$

where L is a "general" (or "generic") proj. plane of dim $N-d$

(Hidden theorem: this number is well-def.)

Next time BKK theorem.