April 6 Zoom

Rem (from previous lectures)

almost every pt. of Y

X is non-sing at

$\mathcal{O}_{p,X}$ $\underbrace{\text{reg. local ring}}_{p \text{ non-sing.}}$ $\Rightarrow$ $\overbrace{\mathcal{O}_{Y,X} \text{ reg. local ring}}$

$\forall\, Y \subset X$ & $p \in Y$

(without proof)

General fact from Comm. alg.:

R reg. local ring $\Rightarrow$ R_p is also reg. local ring.

p prime

Aside Alg. geo. vs. analytic geo.

(check wikipedia page)

Riemann surface = 1-dim compact complex manifold

2-dim over $\mathbb{R}$

. $z \longmapsto z^2$ 2 pts $\longmapsto$ 1 pt function

inverse map 1 pt $\longrightarrow$ 2 pts

$\sqrt{z}$ $a \longmapsto \pm\sqrt{a}$

. $z \longmapsto \exp(z)$ ∞ pts $\longmapsto$ 1 pt

$z + 2\pi k i \longmapsto \exp(z)$

Idea (of Riemann?) (maybe not 1-1)

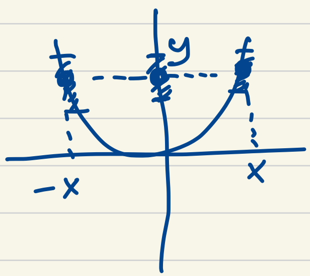
Given (holomorphic) function $f: \mathbb{C} \to \mathbb{C}$

Find a space X s.t. f^{-1} locally is a well-defined 1-1 function.

$f(z) = z^2$

$$X = \{ y - x^2 = 0 \} \subset \mathbb{C}^2$$

$$\begin{array}{ccc} (y,x) & & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ y & \in & \mathbb{C} \end{array}$$



Riemann existence thm:

$\forall$ (compact) Riemann surface X $\exists$ an proj. non-sing. alg. curve X' s.t. $X \cong X'$ as complex manifolds

(i.e. $\forall$ Riemann surface is an alg. curve)

(1950's)

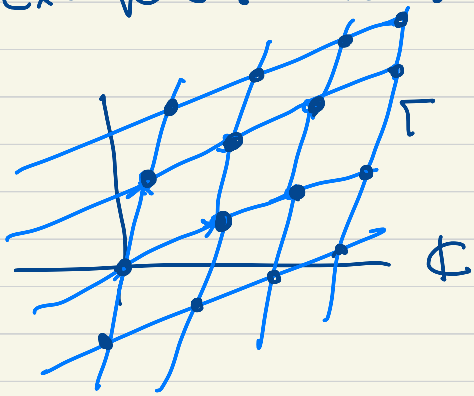
Serre GAGA thm. :

Category of $\boxed{\text{proj.}}$ alg. var. $\xleftrightarrow{\text{equiv.}}$ category of proj. analytic space

 !

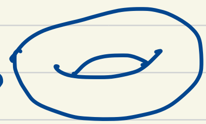
Example: (Chow) Every closed (in Euclidean top.) complex submanifold of complex proj. space is a proj. alg. variety!

Example: 2-dim Torus  as complex manifold (Riemann surface)



$\Gamma \subset \mathbb{C}$

$\Gamma \cong \mathbb{Z}^2$

$\mathbb{C}/\Gamma \rightsquigarrow$ 

$\mathbb{C}/\Gamma \cong$ deg 3 plane curve (in $\mathbb{CP}^2$)

$\{ y^2 = x^3 + ax + b \}$

$\{ y^2 z = x^3 + axz^2 + bz^3 \} \subset \mathbb{CP}^2$

Blow up (end of Sec. I-4)

Blow up $\to$
- top.
- diff. geo. / symp. geo.
- $\boxed{\text{alg. geo.}}$
- comm. alg.

$\mathbb{A}^n \ni 0$ the origin

$$Bl_0(\mathbb{A}^n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$$

(Alg. / alg. set subvarieties in $\mathbb{A}^n \times \mathbb{P}^m$ are defined as zero set of ideals I in $k[x_1, \dots, x_n, y_0, \dots, y_m]$ s.t. I is homog. in the y_j.)

$$Bl_0(\mathbb{A}^n) = \Big\{ \big(\overbrace{(x_1, \dots, x_n)}^{\substack{\text{vec} \\ v}}, \overbrace{(\underbrace{y_1 : \dots : y_n}_{\substack{n \text{ coor.} \\ \text{homog. on } \mathbb{P}^{n-1}}})}^{\substack{\text{line} \\ \ell}} \big) \;\Big|\; v \in \ell \Big\}$$

$v \in \ell \to \exists \lambda \in k \quad (x_1, \dots, x_n) = \lambda (y_1, \dots, y_n)$

Given $(x_1, \ldots, x_n)$ & $(y_1, \ldots, y_n) \neq (0, \ldots, 0)$

when does it happen that :

$(\star)$ $\exists \lambda \in k$ $\boxed{x_i = \lambda y_i}$ $\forall i = 1 \ldots n$?

Suppose $y_j \neq 0$ then $\frac{x_j}{y_j} = \lambda$

Then $x_i = \frac{x_j}{y_j} y_i$ $\forall i$

$\boxed{x_i y_j = x_j y_i}$ $\forall i$ & j s.t. $y_j \neq 0$.

One conclude that $(\star)$ is equiv. to :

$$x_i y_j = x_j y_i \qquad \forall i, j$$

$$\left.\begin{aligned} x_i y_j &= x_j y_i \\ x_k y_j &= x_j y_k \end{aligned}\right\} \xRightarrow{y_j \neq 0} x_i \cancel{y_j} \frac{y_k}{\cancel{y_j}} = x_j y_i \frac{y_k}{y_j}$$

$$= x_k \cancel{y_j} \cdot \frac{y_i}{\cancel{y_j}}$$

$$\Rightarrow x_i y_k = x_k y_i$$

Conclusion: $Bl_0(\mathbb{A}^n)$ = zero set of ideal gen. by $x_i y_j - x_j y_i$ $\forall i, j$.

$\langle x_i y_j - x_j y_i \mid \forall i, j \rangle$ homog. in the y_i.

so $Bl_0(\mathbb{A}^n)$ alg. set in $\mathbb{A}^n \times \mathbb{P}^{n-1}$. ☺

$$\begin{tikzcd} X = Bl_0(\mathbb{A}^n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d, "pr_1"] \arrow[dr, phantom, ""] \\ \mathbb{A}^n \end{tikzcd}$$

$$X = Bl_0(\mathbb{A}^n) \xrightarrow{\varphi} \mathbb{A}^n$$

One checks that ① φ is 1-1 away from $\underline{0}$.

② $\varphi^{-1}(0) = \mathbb{P}^{n-1}$.

$v \in l$

$$(v, [l]) \longmapsto v$$

$v = (x_1, \ldots, x_n)$, $[l] = (y_1 : \cdots : y_n)$

- If $v \neq 0$ then $[l] = [v]$ same pts in $\mathbb{P}^{n-1}$
  i.e. $l =$ line through v & 0.

- If $v = 0$ then $v \in l$ is no condition
  so $\varphi^{-1}(0) = (0, [l])$ $\forall [l] \in \mathbb{P}^{n-1}$

More strongly:

φ is <u>isomorphism</u> from $X \setminus \varphi^{-1}(0)$ to $\mathbb{A}^n \setminus \{0\}$.

$$(x_1, \dots, x_n) \in \mathbb{A}^n \setminus \{0\} \xmapsto{\varphi^{-1}} \big((x_1, \dots, x_n), (x_1 : \dots : x_n)\big)$$

This is a morphism & inverse of φ on $\mathbb{A}^n \setminus \{0\}$.

$$X = Bl_0(\mathbb{A}^n) \xrightarrow{\varphi} \mathbb{A}^n$$

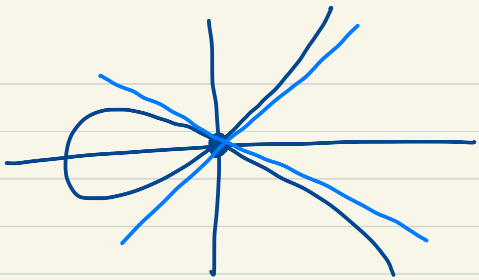
blowup (of affine space at $\underline{0}$).

<u>Idea</u> If $Y \subset \mathbb{A}^n$, $0 \in Y$

Suppose Y is <u>sing</u> at $\underline{0}$.

strict transform of Y $= \overline{\varphi^{-1}(Y \setminus \{0\})}$

Ex.

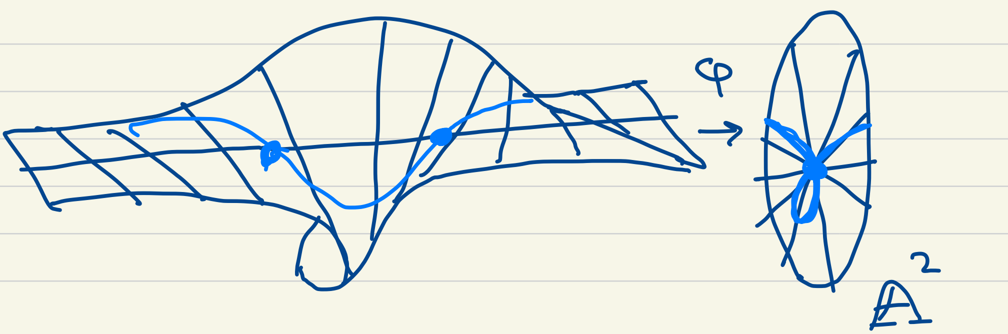


$Y = \{ y^2 = x^2(x+1) \}$

two lines $x=y$
$-x=y$
are tangent cone at 0

(Recall : $\mathbb{CP}^1$ = sphere)

$$\begin{tikzcd}
\tilde{Y} := \overline{\varphi^{-1}(Y \setminus \{0\})} \arrow[d, "\varphi"] & \subset & \mathbb{A}^2 \times \mathbb{P}^1 \arrow[d] \\
Y \arrow[rr, hook] & & \mathbb{A}^2
\end{tikzcd}$$



Exercise see Hartshorne I - Sec. 4

Compute (the equ. of) strict transform $\tilde{Y}$ in this example.

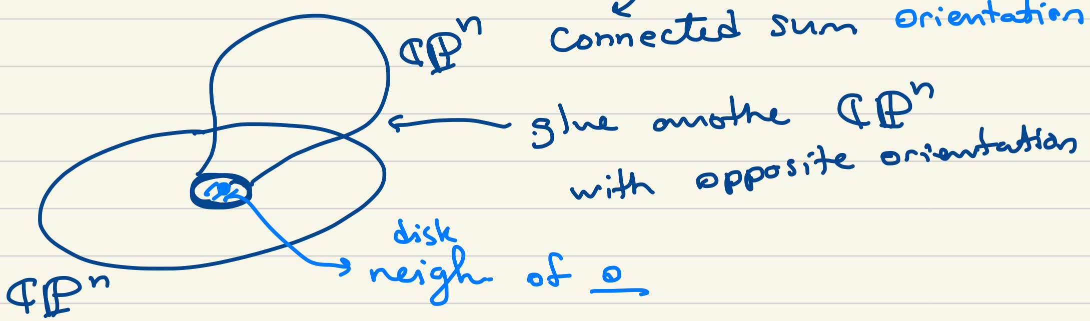
Rem (Top. meaning of blow-up):

(over $k = \mathbb{C}$)

$$Bl_0(\mathbb{CP}^n) = \mathbb{CP}^n \# \overline{\mathbb{CP}^n}$$

$\#$: connected sum

$\overline{\mathbb{CP}^n}$: proj. space with opposite orientation



Rem $Bl_0(\mathbb{A}^n)$ is still a non-sing. var.

Rem (to be explained more next time)

$Y \subset X$ $\quad$ X any var.

Y any subvar.

One can def. $Bl_Y(X)$