April 20 Zoom

People interested in schemes:
Contact Dr. Carl Wang-Erickson (Pitt)
→ Summer course / seminar on schemes

• Hilbert's thm

$X \subset \mathbb{P}^N$ proj. variety with ideal $I = I(X)$

& homog. coor. ring $k[X] := k[x_0, \dots, x_N]/I$.

Let $H_X(m) = \dim_k k[X]_m := k[x_0, \dots, x_N]_m \bmod I$.

Then: ① $\exists$ poly. $P_X(m)$ s.t. $H_X(m) = P_X(m)$ $m \gg 0$.

② $P_X(m)$ is a poly. of degree $r = \dim X$.

③ If $a_r =$ Leading coeff. of $P_X(m)$ then

$d = \deg(X) = r!\, a_r$.

$\deg(X) := |X \cap L|$ L generic plane in $\mathbb{P}^N$ of codim d.

Sketch of proof: → Check Harris's Alg. geo. Sec. 13
Check examples of Hilbert function in there

① Lemma:

Λ generic hyperplane (i.e. codim $\Lambda = 1$) $\dim \Lambda = N-1$

in $\mathbb{P}^N$. defined by $\{\ell = 0\}$

Then ℓ not zero divisor in $k[X]$ &

$\dim(\underbrace{X \cap \Lambda}_{\text{hyperplane section}}) = \dim X - 1$.

Italian school of alg. geo.

- This is Corollary of Bertini thm.

(Famous) Bertini thm. char $k = 0$

$X \subset \mathbb{P}^N$ $\quad \Lambda$ generic hyperplane

- X irr. $\Rightarrow$ $X \cap \Lambda$ irr. $(\dim X > 1)$
- X smooth $\Rightarrow$ $X \cap \Lambda$ smooth
- $(X \cap \Lambda)_{sing} \subset X_{sing} \cap \Lambda$

(2) Let $X' = X \cap \Lambda$ $\quad \Lambda$ generic defined by $l = 0$

$$k[X]_{m-1} \longrightarrow k[X]_m$$

given by multi. by image of l in $k[X]$

l not zero div. $k[X]_{m-1} \hookrightarrow k[X]_m$

$$k[X]_m / \mathrm{Im}(k[X]_{m-1}) \cong k[\underbrace{X \cap \Lambda}_{X'}]_m$$

Note: $\deg X = \deg X'$

One conclude: $H_{X'}(m) = \underbrace{H_X(m) - H_X(m-1)}_{\text{first difference of } H_X(m)}$

(3) Use induction (on $\dim(X)$) i.e. apply ind. hyp. to $X' := X \cap \Lambda$.

Need the base case of induction:

Exercise Suppose X = finite set $\subset \mathbb{P}^N$ $\quad |X| = d \leftarrow \deg(X)$

$\dim X = 0$

$H_X(m) = d$ for $m >> 0$

codim $L = r$
L generic

Rem This also proves $\deg(X) := |X \cap L|$ is well-defined.

Rem Hartshorne I - Sec. 7 defines $\deg X$ as $r! \times$ Leading coeff. of H_X.
(I think this is wrong way of defining degree!)

Rem How big m should be for H_X & P_X to coincide, is related to notions of "regularity" for ideals ... & it is tricky.
(Mumford-Castelnuovo)

Rem Polynomiality of Hilbert function holds in more general situations for f.g. graded modules over a poly. ring.

$k[X]$ $k[x_0, \ldots, x_N]$

Look up Hilbert-Serre thm. (e.g. Hartshorne I-Sec. 7 or Atiyah-Mac.)
(Also $\exists$ related notion of Hilbert-Poincare series)

Introduction to abstract varieties & schemes

- Affine alg. set/var.
  proj. " "
  quasi-proj. " "
  " -aff.

  } reg. map/morphism, reg. function on these

- In analogy with def. of (abs.) manifold → top, real diff., complex diff.

one defines abs. var.:

(see Milne Chap 3 (a))

<u>Def.</u> $(X, \mathcal{O}_X)$ ringed space (top. space + sheaf of algebra)

is an abs. <u>pre</u>variety if:

① X is quasi-compact

② $\forall x \in X \ \exists U \underset{\text{open}}{\subset} X$, $x \in U$,

$(U, \mathcal{O}_X|U) \cong$ ringed space of an affine alg. var. (with structure sheaf)

as ringed spaces

Rem Abs. manifold $\rightsquigarrow$ $\left[\begin{array}{l}\text{2nd. countable} \\ \text{Hausdorff}\end{array}\right.$

replaced with "separation" axiom

replaced with quasi-compact

Lemma Y & Z affine var.

$\varphi_1, \varphi_2 : Z \longrightarrow Y$ morphisms

$\{z \in Z \mid \varphi_1(z) = \varphi_2(z)\} \subset Z$ is closed.

proof WLOG $Z \subset \mathbb{A}^n$, $Y \in \mathbb{A}^m$ &

φ_1 & φ_2 given by polynomials. Then

$\varphi_1 = \varphi_2$ is def. by a finite number of

poly. equalities. ☺

Rem Lemma fails for general prevarieties.

Def. $(X, \mathcal{O}_X)$ prevar. is an abs. var.

if it satisfies separation axiom :

$\forall\, \varphi_1, \varphi_2 : Z \longrightarrow X$ morphism

$\forall\, Z$ affine var.

then $\{ z \in Z \mid \varphi_1(z) = \varphi_2(z) \}$ is closed.

<u>Ex.</u> (standard example of non-separated prevar.)

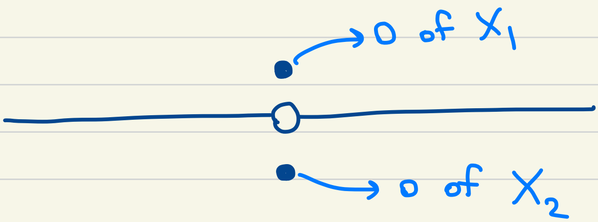
$X_1 = \mathbb{A}^1$

$X_2 = \mathbb{A}^1$

$X^* := X_1 \amalg X_2$

$\sim$ on X^* $\qquad x_1 \sim x_2 \iff x_1 \neq 0$ & $x_1 = x_2$

Define $X := X^* / \sim$



Give X quotient top. $\qquad X_1 \cap X_2 \cong \mathbb{A}^1 \setminus \{0\}$

$X_1 \hookrightarrow X$ & $X_2 \hookrightarrow X$ $\qquad \{X_1, X_2\}$ open cover of X

Define $f : \underset{\text{open}}{U \subset X} \longrightarrow k$ is reg.

$\iff f|_{U \cap X_1}$ & $f|_{U \cap X_2}$ is reg.

. Easy to see X becomes a prevariety.

$$\varphi_1 : \mathbb{A}^1 \hookrightarrow X_1 \subset X$$
$$\shortparallel$$
$$\varphi_2 : \mathbb{A}^1 \hookrightarrow X_2 \subset X$$

Then $\{\varphi_1 = \varphi_2\} = \mathbb{A}^1 \setminus \{0\}$ not closed.

In short: the two origins in X_1 & X_2 can not be separated by regular functions.

- Ex (of abs. var.) Any quasi-proj. variety is an abs. variety.

Because proj. space can be covered with affine space.

<u>Rem</u> One can produce examples of abs. var. which are <u>not</u> quasi-proj. (can not be embedded in some proj. space) (for example $\exists$ non-proj. <u>toric varieties</u>)

$\hookrightarrow$ In contrast with real manifolds: every abs. diff. manifold can be embedded in some $\mathbb{R}^N$

Gluing affine varieties to obtain an abs. variety

Sec. 3.0

(see for example Cox-Little-Schenck Toric varieties)

Suppose we have the following data:

- $\{X_\alpha\}_\alpha$ finite # of affine var.

- $V_{\beta\alpha} \subseteq X_\alpha$ open subset

- $g_{\beta\alpha} : \underset{\subset X_\alpha}{V_{\beta\alpha}} \longrightarrow \underset{\subset X_\beta}{V_{\alpha\beta}}$ iso. of quasi-affine varieties

Satisfying some compatibility conditions:

- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$

- $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$

$$g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha} \quad \text{on} \quad V_{\beta\alpha} \cap V_{\gamma\alpha}$$
$$\forall \alpha, \beta, \gamma$$

Then we can construct an abs. var. out of this data: $X := (\coprod X_\alpha)/\sim$

$x_\alpha \in V_{\beta\alpha} \quad x_\alpha \sim g_{\beta\alpha}(x_\alpha) \in V_{\alpha\beta}$

☺

Next time Proj. construction (example of gluing affine var.)

+ Scheme