April 15 Zoom

- Next week: MW overview of def. of abs. variety & Proj of a graded alg.

  (Projective version of spec. of a f.g. alg.)

- Final: Thursday & Friday takehome exam (hand in by email)

HW4

Problem 1 $C = \{\underbrace{x^6 + y^6 - xy}_{f} = 0\} \subset \mathbb{A}^2$

- $\frac{\partial f}{\partial x}(0,0) = 0 \qquad \frac{\partial f}{\partial y}(0,0) = 0$

  $(0,0)$ sing. pt. $\Rightarrow$ C sing. variety

(1)

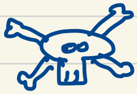
If C normal $\Rightarrow$ $k[C]$ is int. closed

$\Rightarrow$ $\mathcal{O}_{C,p}$ is int. closed $\Rightarrow$ $\mathcal{O}_{C,p}$ is reg. local ring, $\dim \mathcal{O}_{C,p} = 1$

$p = (0,0)$

$\Rightarrow$ p is non-sing Contradiction

So C not normal.



$p=(0,0)$

(2) If m_p principal $\Longrightarrow$ $\mathcal{O}_{C,p}$ is regular.

I-

(3) Compare with Ex. 4.9.1 in Hartshorne

$y^2 = x^2(x+1)$

$\tilde{C} = \overline{\phi^{-1}(C\setminus 0)}$ $\quad$ $\phi: Bl_0(\mathbb{A}^2) \longrightarrow \mathbb{A}^2$

Basically points in $\tilde{C}$ which lie over 0

are $(0,[l])$ s.t. l is a tangent line to C at $\underline{0}$
$0 \in l$

(l belongs to tangent Cone of C at 0)

$f = x^6 + y^6 - xy = 0 \rightsquigarrow$ Equ. of tangent Cone at $\underline{0}$ is:
Equ. of C

$xy = 0$

(degree 2 = lowes deg. part of f)

$xy = 0 \Longrightarrow x=0$ or $y=0$ two points lying over $\underline{0}$.

Problem 2 Showing Completion of (pt=(0,1))
local ring of $k[x,y]/(x^2+y^2-1)$ at max ideal $(x, y-1)$

$m_p = (x, y-1)$

$A = \left(k[x,y] \big/ (x^2+y^2-1) \right)_{m_p}$

you can use def.

$$\hat{A}_I := \varprojlim A/I^n$$

$I = m_p$

to show $\hat{A}_I \cong k[[t]]$.

One also can use the general fact that: $A = R/J \rightsquigarrow \hat{A} = \hat{R}/\hat{J}$

$\hat{J}$ = ideal in $\hat{R}$ gen. by $J \subset R$

For simplicity $u = y-1$

$u+1 = y$

$$A = \left(k[x,u] \big/ (x^2+(u+1)^2-1) \right)_{(x,u)}$$

$$\hat{A} = k[[x,u]] \big/ (x^2+(u+1)^2-1) \cong k[[u]]$$

$x^2 = 1-(u+1)^2$ has a sol. $x = \pm$ power series in u

Proof of BKK Cont'd

$k[x_1^{\pm}, \dots, x_n^{\pm}, t]$

Let $R(A) = k[1, x^{\alpha_0} t, \dots, x^{\alpha_N} t] \subset$

Consider graded k-alg. homomorphism $k[z_0, \dots, z_N] \xrightarrow{\psi} R(A)$

$z_i \overset{\psi}{\longmapsto} x^{\alpha_i} t$

$1 \overset{\psi}{\longmapsto} 1$

ψ is clearly surjective & preserves grading

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deg m part of $R(A)$ =

$$\left\{ \text{Lin. combs of } \underbrace{(x^{\alpha_0} t)^{m_0} \cdots (x^{\alpha_N} t)^{m_N}}_{\prod_{j=0}^{N} x^{\alpha_j m_j} \, t^m} \;\middle|\; \begin{array}{c} m_0, \dots, m_N \geq 0 \\ m_0 + \dots + m_N = m \end{array} \right\}$$

$$\left\{ x^{\sum_j m_j \alpha_j} \;\middle|\; \begin{array}{c} m_0, \dots, m_N \geq 0 \\ \sum m_j = m \end{array} \right\}$$

Recall: $A + B := \{\alpha + \beta \mid \alpha \in A, \beta \in B\}$

$$\left\{ x^{\alpha} \;\middle|\; \alpha \in m * A := \underbrace{A + \dots + A}_{m \text{ times}} \right\}$$

Conclusion: $\dim_k R(A)_m = |m * A|$.

as graded alg.

But $k[X_{\mathcal{A}}] \cong R(\mathcal{A})$

So $H_{X_{\mathcal{A}}}(m) = |m * \mathcal{A}|$

(recall $H_X(m) := \dim_k k[X]_m$)

$X \hookrightarrow \mathbb{P}^N$

any proj. var. $X \subset \mathbb{P}^N$

Recall Hilbert's thm:

① $\exists\, P_X$ poly. in m s.t. $H_X(m) = P_X(m)$ $m \gg 0$

② $\deg P_X = \overbrace{\dim X}^{r}$ (as variety)

③ $P_X(m) = a_r m^r +$ lower terms

then $\deg(X \hookrightarrow \mathbb{P}^N) = r!\, a_r$.

Combinatorial problem:

Fix a finite subset $\mathcal{A} \subset \mathbb{Z}^n$

show for $m \gg 0$ $|m * \mathcal{A}|$ is a poly. in m.

Hilbert fnction

<u>proof</u> $|m * A| = H_{X_A}(m) \quad \forall m \in \mathbb{N}$

Then use Hilbert's thm.

$\boxed{\text{Next Comb problem:}}$ Suppose $\Delta = \text{Conv}(A)$ & $0 \in A$ & A generates $\mathbb{Z}^n$

$$|m * A| \approx \underbrace{\text{vol}_n}(\Delta)\, m^n$$

usual Lebesgue measure

Riemann sum approx for vol.

The proof is by showing:

$$|m * A| \approx |\underbrace{m\Delta \cap \mathbb{Z}^n}_{\text{lattice pts in } m\Delta}|$$

Ehrhart function/poly. of Δ

Note: $A \subset \Delta \cap \mathbb{Z}^n$

$m * A \subset m\Delta \cap \mathbb{Z}^n$

$|m * A| \leqslant |m\Delta \cap \mathbb{Z}^n|$

proof is elementary (ask me for a reference)

Putting everything together:

$$\deg(X_A) = r!\, a_r$$

leading coeff. of $P_{X_A}(m) = |m*A|$
$m \gg 0$

If $0 \in A$, A gen. $\mathbb{Z}^n$ $\rightsquigarrow$ $\Phi_A : (k\setminus 0)^n \longrightarrow \mathbb{P}^N$

is one-to-one & hence $\dim X_A = n$

so $\deg X_A = n!\ \mathrm{vol}_n(\Delta)$.

Finally, $\deg X_A = \left| \mathrm{Im}\,\Phi_A \cap L \right|$

L: generic plane of codim n

because $\mathrm{Im}\,\Phi_A$ contains an open set

& RHS by def. is $\underbrace{[\mathcal{L}_A, \dots, \mathcal{L}_A]}_{\#\text{ of sol. of a generic system from } \mathcal{L}_A}$