April 13 Zoom

Back to BKK thm.

$A = \{\alpha_0, \dots, \alpha_N\} \subset \mathbb{Z}^n$

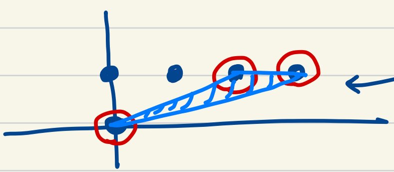
$k[x_1^{\pm}, \dots, x_n^{\pm}]$

$$\mathcal{L}_A = \Big\{ f = \sum_{\alpha \in A} c_\alpha x^\alpha \;\Big|\; c_\alpha \in k \Big\} \subset$$

Ex. $n = 2$ (x, y) variables

$A = \{(0,0), (2,1), (3,1)\}$

Conv(A) = triangle



$$\mathcal{L}_A = \{ \square + \square x^2 y + \square x^3 y \}$$

generically

BKK thm: Pick $a_1, a_2, a_3, b_1, b_2, b_3 \in k$

$\#$ $(x, y) \in (k \setminus 0)^2$ s.t.

i.e. $x \neq 0$ & $y \neq 0$

$$\begin{cases} a_1 + a_2 x^2 y + a_3 x^3 y = 0 \\ b_1 + b_2 x^2 y + b_3 x^3 y = 0 \end{cases}$$

is $2!\,\text{Area}(\text{conv}(A)) = 2!\,\frac{1}{2} = 1.$

not spanned by monomials

Aside (# of sol. of arbitrary systems)

X any variety $n = \dim X$

$k(X)$ field of rat. functions

Fix $\mathcal{L} \subset k(X)$ finite dim k-vec. subspace

Then: (K. - Khovanskii, 2010) Theory of Newton-Okounkov bodies

Thm

① $\exists$ non-empty open $V \subset X$ $\exists$ non-empty open $\mathcal{U} \subset \underbrace{\mathcal{L} \times \cdots \times \mathcal{L}}_{n \text{ times}}$

$(f_1, \ldots, f_n) \in \mathcal{U}$ (generic system from L)

$$\left|\{x \in V \mid f_1(x) = \cdots = f_n(x) = 0\}\right|$$

is ind. of $(f_1, \ldots, f_n)$.

② This (# of sol. of a generic system from $\mathcal{L}$)

is equal to $n!\, \mathrm{vol}_n(\Delta)$.

for some convex body (Convex + Compact) $\Delta = \Delta(X, \mathcal{L}) \subset \mathbb{R}^n$.

<u>Rem</u> # of sol. of a generic system

is finite & const. (ind. of choice of system)

is an <u>algebraic</u> effect (not true in real or complex analysis)

<u>Ex.</u> $k = \mathbb{R}$ $\sin(x) + 1 = 0$ ∞ many sol.

$\sin(x) + 2 = 0$ no sol.

$k = \mathbb{C}$ $e^x = 2 \rightsquigarrow$ ∞ many sol.

$x \rightsquigarrow x + 2k\pi i$
sol. $k \in \mathbb{Z}$ sol.

A notation: $A \subset \mathbb{Z}^n$ finite subset

$[\mathcal{L}_A, \dots, \mathcal{L}_A] =$ # of sol. of a gen. system $f_1(x) = \dots = f_n(x) = 0$

We would like to realize $[\mathcal{L}_A, \dots, \mathcal{L}_A]$

as deg. of some proj. variety $X_A \subset \mathbb{P}^N$

$|A| = N + 1$.

$$x = (x_1, \dots, x_n) \longmapsto (x^{\alpha_0} : \dots : x^{\alpha_N})$$

$$(k \setminus 0)^n \xrightarrow{\Phi_A} \mathbb{P}^N$$

Ex. $A = \{ (0,0), (2,1), (3,1) \}$

$$(k \setminus 0)^2 \longrightarrow \mathbb{P}^2$$

$$(x,y) \longmapsto (1 : x^2y : x^3y)$$

Let $X_A :=$ closure of image of Φ_A in $\mathbb{P}^N$

($\mathrm{Im}(\Phi_A)$ not closed in $\mathbb{P}^N$)

e.g. $(0:0:1) \in X_A$ but $\notin \mathrm{Im}(\Phi_A)$

$x = y \to \infty$ $\quad (1 : x^2y : x^3y) \overset{\text{div. by } x^4}{=}$

$$\left(\frac{1}{x^4} : \frac{1}{x} : 1 \right) \xrightarrow[x \to 0]{} (0:0:1)$$

$x = y \to 0$ $\quad (1:0:0)$

I think in this ex. Φ_A is injective &
$X_A = \mathbb{P}^2$

Rem (General statement about closure of image of a morphism)

$\Phi : X \longrightarrow Y$ dominant morphism i.e. $\Phi(X)$ dense in Y

(Hopefully I prove next time if there is time)

Thm & it is a Cor. of Noether normalization

$\mathrm{Im}\,\Phi$ Contains a non-empty open subset of Y.

In previous situation:

$$\Phi_A : \underbrace{T = (k \setminus 0)^n}_{\text{algebraic torus}} \longrightarrow \underbrace{X_A}_{\text{proj. subvar. of } \mathbb{P}^N} \subset \mathbb{P}^N$$

$\mathrm{Im}(\Phi_A)$ Contains an open subset of X_A.

(Toric variety)

Def. X_A is a Proj. toric variety.

(ass. to the finite of lattice pts $A \subset \mathbb{Z}^n$).

homog. coor. ring

We would like to describe $k[X_A]$

& $\deg(X_A)$.

keeps track of grading

<u>Thm</u>

(i) $k[X_A] \cong$ k-subalg. of $k[x_1^{\pm}, \dots, x_n^{\pm}, t]$

as graded k-alg.

$$= \bigoplus_{m \geq 0} k[x_1^{\pm} \dots x_n^{\pm}]\, t^m$$

gen. by $\underbrace{1}_{\deg = 0}$ & $\underbrace{x^{\alpha_0} t, \dots, x^{\alpha_N} t}_{\deg 1}$

(ii) $\deg(X_A) = [L_A, \dots, L_A]$

$k[x_1^{\pm} \dots x_n^{\pm}, t]$

<u>proof of (i)</u>

Let $R(A) = k[1, x^{\alpha_0} t, \dots, x^{\alpha_N} t] \subset$

Consider graded k-alg. homo<u>ism</u> $k[z_0, \dots, z_N] \xrightarrow{\psi} R(A)$

$z_i \xmapsto{\psi} x^{\alpha_i} t$

$1 \xmapsto{\psi} 1$

ψ is clearly surjective & preserves grading let's show $\ker \Psi$ is $I = I(X_A)$.

Then we get iso. $k[X_A] := k[z_0, \dots, z_N] / I(X_A)$

$\cong R(A)$.

$f = \sum_{\substack{\beta = (\beta_0, \dots, \beta_N) \\ \sum \beta_j = m}} c_\beta z^\beta$ homog. poly. in I of deg m

f is in $\ker \Psi \iff \sum_{\substack{\beta = (\beta_0, \dots, \beta_N) \\ \sum \beta_j = m}} c_\beta (x^{\alpha_0} t)^{\beta_0} \cdots (x^{\alpha_N} t)^{\beta_N} = 0$

$\iff \sum_\beta c_\beta x^{\alpha_0 \beta_0 + \cdots + \alpha_N \beta_N} = 0$

$\iff f|_{\operatorname{Im} \Phi_A} = 0$

because $\operatorname{Im} \Phi_A$ is dense in X_A

$\iff f|_{X_A} = 0$

$\iff f \in I(X_A)$.

Idea of proof of BKK thm.

$$R(A)_m = \mathrm{span}\left\langle x^\alpha \;\middle|\; \begin{array}{l}\alpha = m\text{-fold} \\ \text{sum of} \\ \alpha_0, \ldots, \alpha_N\end{array} \right\rangle$$

$$\dim R(A)_m = |\underbrace{A + \cdots + A}_{m \text{ times}}|$$

$$\underbrace{A + \cdots + A}_{m * A} = \{ \gamma_1 + \cdots + \gamma_m \mid \gamma_i \in A \}$$

- Lemma: $|m * A| \approx \mathrm{vol}_n(\Delta)\, m^{\dim \Delta}$

$$\Delta = \mathrm{conv}(A).$$

$\to$ pure combinatorics!

BKK then follows from Hilbert's thm.