April 10 Zoom

One the main goals of intersec. theory is the following: Given variety X (proj. or complete)

Construct a <u>ring</u> structure on subvarieties of X (similar to cohomology ring in topology)

$+ \rightsquigarrow$ Formal addition

alg. cycle $= \sum_i a_i Z_i$

<u>finite</u> formal sum of subvar. $Z_i \subset X$ & a_i from some coeff. ring. such as $\mathbb{Z}$ or $\mathbb{Q}$.

$\times \rightsquigarrow$ Intersection of subvarieties

For an intro/review see appendix in Hartshorne.

- Main theorem: this is doable when X is non-sing. (called <u>Chow ring of X</u>)

Degree of a proj. var. $X \subset \mathbb{P}^N$

Assumption: char $k = 0$ $\overline{k} = k$ (if you like $k = \mathbb{C}$)

Def. / thm Let $r = \dim X$

Consider space of all $(N-r)$-dim proj. planes in $\mathbb{P}^N$ (image of all $(N+1-r)$-dim. planes in $\mathbb{A}^{N+1}$)

This is Grassmannian $Gr(N-r, N)$.

(one shows $Gr(N-r, N)$ is itself a proj. variety)

$L = V(\langle l_1, \dots, l_r \rangle)$ lin. ind. l_i lin. poly. in $x_0, \dots, x_N$

$L \in Gr(N-r, N)$

$\tilde{\mathcal{U}} \overset{\text{Zariski open}}{\subset} \underbrace{k^{N+1} \times \dots \times k^{N+1}}_{r}$ of all $(l_1, \dots, l_r)$ $\cong \mathrm{Mat}_{r \times N+1}(k)$

that are lin. ind.

$\tilde{\mathcal{U}} \twoheadrightarrow Gr(N-r, N)$

depending on X

$\exists\, \mathcal{U} \subset \tilde{\mathcal{U}} \subset \mathrm{Mat}_{r \times N+1}$

Zariski open
non-empty

(or equiv. you can take a Zariski open in $Gr(N-r, N)$).

such that

If $(\ell_1, \dots, \ell_r) \in \mathcal{U}$ & $L = V(\langle \ell_1, \dots, \ell_r \rangle) \in \mathbb{P}^N$

codim r proj. plane

L is "generic" or in "general position"

then $|X \cap L|$ is finite & ind. of choice of L.

cardinality of this set

We call this number degree of X.

$d = \deg(X)$

Rem V some variety, any $x \in V$ is called "generic" or "in general position" if $x \in \mathcal{U} \subset V$.

open
non-empty

Ex. $X \subset \mathbb{P}^N$ $\dim X = 0$

X finite set

$|X| = d$

$\deg(X) = ?$ $\dim X = \operatorname{codim} L = 0$

$\dim L = N$

$\underbrace{Gr(N, N)}_{\text{only one element}} = \{\mathbb{P}^N\}$ ← only possible L

$\deg X = |X \cap \mathbb{P}^N| = |X| = d.$

Ex. $X \subset \mathbb{P}^N$ X r-dim proj. plane

L $N-r$-dim proj. plane → generic

$|X \cap L| =$

$\tilde{X} \subset \mathbb{A}^{N+1}$ $\tilde{L} \subset \mathbb{A}^{N+1}$

affine cone

$\dim \tilde{X} = r+1$ $\dim \tilde{L} = N+1-r$

$\dim \tilde{X} \cap \tilde{L} = 1$ $X \cap L = \{pt\}$

line through $\underline{0}$ $|X \cap L| = 1.$

Ex. $X = V(f) \subset \mathbb{P}^2$

$f(x,y,z)$ homog. poly. of deg d.

$\deg(X) = d$ $\quad$ $\dim X = 1$

$\dim L = 1$

$\hookrightarrow$ proj. line

$$\boxed{|X \cap L| = d}$$

$$L = \{ (x:y:z) \mid \ell_1 x + \ell_2 y + \ell_3 z = 0 \}$$

$$z = -\frac{\ell_1}{\ell_3} x - \frac{\ell_2 y}{\ell_3}$$

sub-in $\quad$ in $f(x,y,z) = 0$

we get homog. poly. in x, y of deg d.

$\hookrightarrow$ If ℓ_1, ℓ_2, ℓ_3 generic it has exactly d solutions $\mathbb{A}^2 \subset \mathbb{P}^2$ by assumption that k alg. closed

& char. $k = 0$.

- Same works for hypersurf.

$$X = V(f) \subset \mathbb{P}^N$$

f = homog. poly. of degree d.

<u>Rem</u> You can define degree of an affine variety $X \subset \mathbb{A}^N$

$$\begin{array}{ccc} X & \subset & \mathbb{A}^N \\ \wr & & \cap \\ \overline{X} & \subset & \mathbb{P}^N \end{array}$$

<u>Rem</u> Lots of problems in geo./alg./comb. are about finding degree of a variety.

<u>BKK theorem</u> $\rightsquigarrow$ Bernstein – Kushnirenko – Khovanskii

(1970's)

- It is about number of sol. of a system of poly. equ. with fixed exponents.

$\alpha = (a_1, \dots, a_n) \in \mathbb{Z}^n$

$x = (x_1, \dots, x_n)$

$x^\alpha := x_1^{a_1} \cdots x_n^{a_n}$ a monomial in the x_i

Fix a finite set $A = \{\alpha_0, \dots, \alpha_N\} \subset \mathbb{Z}^n$

Consider the vec. space

$$\mathcal{L} = \left\{ f \in k[x_1^{\pm}, \dots, x_n^{\pm}] \;\middle|\; f = \sum_{i=0}^{N} c_i x^{\alpha_i} \right\}$$

$$= \operatorname{Span}\{x^{\alpha_0}, \dots, x^{\alpha_N}\}$$

$$\mathcal{L} \subset k[x_1^{\pm}, \dots, x_n^{\pm}]$$

- Any $f \in k[x_1^{\pm} \dots x_n^{\pm}]$ can be evaluated at $(k \setminus 0)^n$.

Exercise: $T = (k \setminus 0)^n \subset \mathbb{A}^n$ open Zariski

T → alg. torus

(quasi-affine by def.)

is itself affine var. (i.e. iso. to an affine var.)

$n=1$:

$\mathbb{A} \setminus 0 \subset \mathbb{A}$

$\{x \neq 0\} \cong \{xy - 1 = 0\} \subset \mathbb{A}^2$

$$\underbrace{k[T]}_{} \cong k[x_1^{\pm}, \dots, x_n^{\pm}].$$

↳ ring of reg. functions on T

For Lingyu:

Exercise: $\mathbb{A}^n \setminus \{0\}$ is quasi-affine but not affine var.

$$\mathcal{O}(\mathbb{A}^n \setminus \{0\}) = \mathcal{O}(\mathbb{A}^n) = k[x_1 \cdots x_n].$$

BKK thm

Fix $A = \{\alpha_0, \dots, \alpha_N\} \subset \mathbb{Z}^n$

$$\mathcal{L} = \mathcal{L}_A = \operatorname{span}\{x^{\alpha_0}, \dots, x^{\alpha_N}\} \subset k[x_1^{\pm}, \dots, x_n^{\pm}]$$

$$\dim_k \mathcal{L} = N+1 = |A|.$$

coeff. c_i in $\sum c_i x^{\alpha_i}$ are random

Take $(f_1, \dots, f_n)$ "generic" $\in \underbrace{\mathcal{L} \times \dots \times \mathcal{L}}_{n}$

Then
$$\left| \left\{ z = (z_1, \dots, z_n) \in (k \setminus 0)^n \;\middle|\; f_1(z) = \dots = f_n(z) = 0 \right\} \right|$$

is: (1) ind. of $(f_1, \dots, f_n) \in \mathcal{L} \times \dots \times \mathcal{L}$

(2) This number is equal:

$$n! \, \underbrace{\mathrm{Vol}_n}\big(\mathrm{Conv}(A) \big).$$

usual Euclidean vol. (Lebesgue measure)

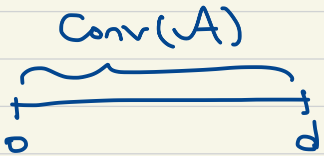
Ex. $n=1$ $A=\{0,\dots,d\}$

$\mathcal{L}$ = All poly. of degree d.

BKK: If coeff. of $f(x)=c_0+c_1x+\dots c_dx^d$ are "generic" (f gen. poly. of deg d)

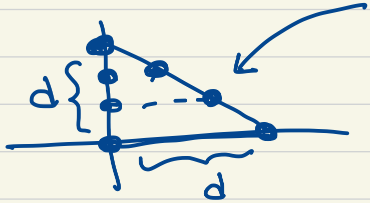
then # of roots of f $=!!\ d$

$= d$ ☺



length of $\mathrm{Conv}(A) = d$

Ex. $n=2$



A = lattice pts in this triangle

$\mathcal{L}$ = All poly. of deg. d in x,y

$f, g \in \mathcal{L}$ generic

$$x, y \neq 0$$

$$f(x,y) = g(x,y) = 0$$

$$\#\text{ of sol.} = 2!\ \underbrace{\text{Area of } \triangle}_{\frac{1}{2}d^2}$$

$$= d^2 \longrightarrow \text{agrees with Bezout thm.}$$

Proof uses the important notion of Hilbert function / polynomial of a proj. var. / graded module.

$X \subset \mathbb{P}^N$ proj. var.

$I = I(X)$ homog. ideal.

$$k[X] = \text{homog. coor. ring} := k[z_0, \ldots, z_N] / I$$

$$k[X] = \bigoplus_{m \geq 0} k[X]_m$$

$$k[X]_m = k[z_0, \ldots, z_N]_m \mod I$$

$$\dim k[X]_m < \infty$$

Def. $H_X : \mathbb{N} \longrightarrow \mathbb{N}$

Hilbert function of $X \hookrightarrow \mathbb{P}^N$

is : $H_X(m) := \dim_k k[X]_m$.

Thm (Hilbert) Let $r = \dim X$

- $\exists$ poly. $P_X(m)$ s.t.

  $H_X(m) = P_X(m)$ for $m \gg 0$.

- degree of poly. $P_X(m) = \dim X$

- degree of $X = r! \cdot$ leading coeff. of P_X