April 1 Zoom

Recall:

X irr. affine var. $A = k[X]$

$\overline{A} = \overline{k[X]}$ = int. closure of $k[X]$ in $k(X)$.

(by Noether's Finiteness of int. closure $\overline{A}$ is a f.g. A-module & hence itself a f.g. alg.)

$i: A \hookrightarrow \overline{A} \Longrightarrow \pi : \tilde{X} \longrightarrow X$

① π surj. finite map.

② π birat. isoism.

Def. X irr. var. is a normal variety if all local rings $\mathcal{O}_{P,X}$ are int. closed (in $k(X)$).

(If X affine, this is equiv. to $k[X]$ be int. closed in $k(X)$).

- If $\dim X = 1$ then X is normal $\iff$ X is non-sing.

- In general, suppose X normal. Then $\forall Y \subset X$, $\operatorname{codim} Y = 1$, Y irr. then $\mathcal{O}_{Y,X}$ is a reg. local ring (hence a DVR). because $\dim \mathcal{O}_{Y,X} = \operatorname{codim} Y = 1$

non-sing. in codim 1

Cor. (without proof) X normal $\Longrightarrow$ Codim $X_{sing} \geq 2$.

Rem X affine irr. curve over $k=\mathbb{C}$; $k[X]$ coor. ring.

Let $f \in k(X)$ rat. function. Then

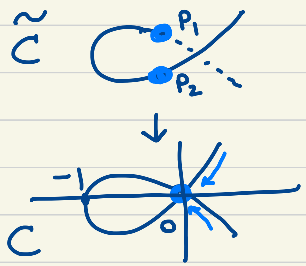
$f \in \overline{k[X]}$ (i.e. integral over $k[X]$)

$\Longleftrightarrow$ f is bounded near every $p \in X$
(it maybe undefined at p)

proof of ($\Longrightarrow$) : $f^m + a_{m-1} f^{m-1} + \dots + a_0 = 0$

Suppose $y_i = f(x_i) \longrightarrow \infty$ as $x_i \longrightarrow p$.

But $|y_i| \leq |a_{m-1}| + |a_{m-2} \frac{1}{y_i}| + \dots + |a_0 \frac{1}{y_i^{m-1}}|$ $\rightsquigarrow$ bounded (∵ a_i's bounded, $\frac{1}{y_i} \to 0$) ☺



$\frac{y}{x} - 1$ vanishes at P_2
but not on P_1.

Ex. $C = \{y^2 = x^2(x+1)\}$ Sing. at $(0,0)$

$k = \mathbb{C}$

rat. function, not reg. function which is int. over $k[C]$

$$f(x,y) = \frac{y}{x}$$

$$\left(\frac{y}{x}\right)^2 - (x+1) = 0$$

$$\lim_{\substack{(x,y) \to (0,0) \\ \text{on } C}} \frac{y}{x} = \begin{cases} +1 \\ -1 \end{cases} \quad \text{DNE}$$

$$v_p(f) = i \quad \text{where} \quad f \in m_p^i, \quad f \notin m_p^{i+1}$$

$$m_p \supset m_p^2 \supset \cdots$$

(Krull's intersec. thm. : $\bigcap_{i \geq 0} m_p^i = \{0\}$)

Def. - X variety, $D \subset X$ is "prime divisor" is an irr. subvar. of X of codim 1.

historically because of parallel stories of coor. rings of curves & rings of int. in number theory

- A "divisor" D in X is a formal finite linear comb.

$$D = \sum_i a_i D_i \qquad D_i \text{ prime div.}, \quad a_i \in \mathbb{Z}$$

- Suppose X normal variety, $D \subset X$ prime div. $f \in k(X)$ rat. function

Then $v_D(f)$ = order of vanishing of f along D

It is the val. ass. to the reg. local ring of dim 1 $\mathcal{O}_{D,X}$. DVR

Ex (stupid but useful)

$X = \mathbb{A}^2 \qquad f(x,y) = x^2y^2 + xy^3$

$D_1 = V(x) \rightsquigarrow$ y-axis

$D_2 = V(y) \rightsquigarrow$ x-axis

$v_{D_1}(f) = 1 \rightsquigarrow x(xy + y^3)$

$v_{D_2}(f) = 2 \rightsquigarrow y^2(x^2 + xy)$

<u>Rem</u> $D \subset X$ prime div. & $\mathcal{O}_{D,X}$ reg. local ring ($=$ DVR) means that D has a "local equ."

$\exists\, u \in \mathcal{O}_{D,X} \qquad u : \underset{\substack{\text{open} \\ \text{in } X}}{U} \longrightarrow k \qquad U \cap D \neq \emptyset$

s.t. $D \cap U = V(u)$

more strongly, $\underset{\text{max ideal}}{\mathfrak{m}_{D,X}} = (u)$.

So $\forall f \in k(X)$ near D, $f = u^a g$

$a = v_D(f)$ & $\underset{\in \mathcal{O}_{D,X}}{g}$ does not vanish on D.

Ex. (of $\mathcal{O}_{D,X}$) $\quad X = \mathbb{A}^2$

$D = V(Y)$

$f = \frac{y}{x} \in \mathcal{O}_{D,X}$ because $\frac{y}{x}$ is reg.

on the open set $U = \{x \neq 0\}$ & $\underset{x\neq 0}{U} \cap \underset{y=0}{D} \neq \emptyset$.

$\frac{y}{x}$ vanishes on $D \cap U$ of order 1.

($\frac{y}{x}$ undefined at origin)

Brief discussion of birational geo. of curves

(Sec. I-6 in Hartshorne)

(k base field)

- Fix a field of dim 1 K.

tr deg $K/k = 1 \rightsquigarrow K/k(x)$ finite ext.

Question ① $\exists$ irr. curve C s.t. $k(X) \cong K$?

normalization / res. of sing. of curve

② Given curve C with $k(C) = K$

$\exists C'$ $\quad C'$ non-sing. & $k(C') = K$?

we skip, see Hartshorne

Thm. $\exists$ 1-1 corr. between

non-sing. proj. curve up to isoism $\xleftrightarrow{1-1}$ curves up to birational isoism

$\xleftrightarrow{1-1}$ K/k field ext. of dim 1 $\xleftrightarrow{1-1}$ (easy)

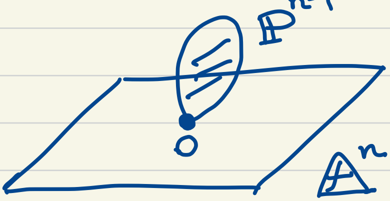
quasi-proj. curve

$C \rightsquigarrow \tilde{C}$ proj. non-sing.

(non-sing. model of C)

Blow up I-4 Hartshorne

$$Bl_0(\mathbb{A}^n) \subset \mathbb{P}^{n-1} \times \mathbb{A}^n$$



$$Bl_0(\mathbb{A}^n) \xrightarrow{\pi} \mathbb{A}^n$$

birational morphism