MATH 2810 Algebraic Geometry, Homework 3

Kiumars Kaveh

April 1, 2020

Due: Monday April 13, 2020

- In all the problems $\mathbf{k}$ denotes the ground field and is assumed to be algebraically closed. If not specified, by an algebraic variety we mean a quasi-projective algebraic variety. You can of course use theorems stated in class.

Problem 1: Let $C \subset \mathbb{A}^2$ be the curve defined by the irreducible polynomial

$$x^6 + y^6 - xy = 0.$$

(1) Show that C is not a normal variety. (2) Show that the maximal ideal of the origin $o \in C$ is not a principal ideal. (3) Let $\phi : \mathrm{Bl}_o(\mathbb{A}^2) \to \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at the origin o. Let $\tilde{C}$ be the strict transform of C, i.e. $\tilde{C} = \overline{\phi^{-1}(C \setminus \{o\})}$. Describe the points in $\tilde{C}$ which lie above $o \in C$.

Problem 2: Consider the circle $C = V(x^2+y^2-1)$ and the line $L = V(y-1)$. Consider the point $p = (0, 1)$ which lies on both C and L and let $R = \mathcal{O}_{p,C}$ and $R' = \mathcal{O}_{p,L}$ be local rings of C and L at p respectively. Verify the Cohen Structure Theorem directly by showing that the completions $\hat{R}$ and $\hat{R}'$ are both isomorphic (as $\mathbf{k}$-algebras) to formal power series ring in one variable.

Problem 3: For simplicity let $\mathbf{k} = \mathbb{C}$. Consider the affine curves C below (in $\mathbb{A}^2$) with given points on them. (1) If p is a non-singular point, verify directly that the maximal ideal in the corresponding local ring is principal by finding a single generator for it. (2) If p is a singular point verify directly that the corresponding local ring is not integrally closed.

(a) $y^3 = x^4$, $p = (0, 0)$.

(b) $x^3 + y^3 = 1$, $p = (1, 0)$.

Problem 4: Show that the cone $x^2 + y^2 = z^2$ is a normal variety, even though the origin is singular (characteristic $\neq 2$).

Other problems (no need to hand in)

Problem: Given a smooth point on a variety and a tangent vector at the point, show that there is a smooth curve passing through the point with the given vector as its tangent vector.