MATH 2810 Algebraic Geometry, Homework 3

Kiumars Kaveh

March 14, 2020

Due: Monday March 30, 2020

- In all the problems **k** denotes the ground field and is assumed to be algebraically closed. If not specified, by an algebraic variety we mean a quasi-projective algebraic variety.

Problem 1: Let X be an (quasi-projective) algebraic variety and let $p \in X$. Show that $\mathcal{O}_{p,X}$ is an integral domain if and only if p belongs to a unique irreducible component of X (hint: first reduce the problem to the case where X is an affine variety).

Problem 2: Recall that the field of rational functions $\mathbf{k}(X)$, of an irreducible (quasi-projective) algebraic variety X, is the collection of all rational functions on X. A rational function is a regular function f on some non-empty open subset $U \subset X$ (up to the equivalence that $(f, U) \cong (g, V)$ if $f = g$ on $U \cap V$.

(a) Verify that any two open subsets in X intersect and hence the field operations on $\mathbf{k}(X)$ are well-defined. Then show that $\mathbf{k}(X)$ is in fact a field.

(b) Show that if X is an affine variety then its field of rational functions coincides (is naturally isomorphic to) the field of fractions of its coordinate ring $\mathbf{k}[X]$ (you can use or refer to material from class).

Problem 3: Show that any two smooth quadrics in $\mathbb{P}^n$ are isomorphic. Recall that a quadric (in $\mathbb{P}^n$) is a subvariety defined by a (homogeneous) quadratic polynomial.

Problem 4: Consider the affine curves C (in $\mathbb{A}^2$) below. Find the points at infinity on these curves (that is, points on the closure of C in $\mathbb{P}^2$ that are not in $\mathbb{A}^2$). Decide for each point at infinity if it is singular or non-singular and find its tangent space and tangent cone.

(a) $y^2 = x^3 + ax + b$

(b) $y = x^3$

(c) $x^3 + x^2y - y = 0$

Other problems (no need to hand in)

Problem: Show that the Plücker map from $\mathrm{Gr}(2,4)$ to $\mathbb{P}^5$ is an embedding and the image is a closed subvariety (it is in fact, a hypersurface given by one equation, the Plücker relation).