MATH 2810 Algebraic Geometry, Homework 3

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April 10, 2015

Due: April 20, 2015

In all the problems the ground field $\mathbf{k}$ is an algebraically closed field. In some of the problems the ground field is the field of complex numbers $\mathbb{C}$.

Problem 1: Let $\mathbf{k} = \mathbb{C}$. Show that the projective plane $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent (very easy). On the other hand, show that they are not isomorphic varieties (for example you can use algebraic topology, you are allowed to use statements from topology without proof).

Problem 2: Consider the polynomial algebra $S = \mathbf{k}[x_0, \dots, x_n, y_0, \dots, y_m]$. We say that an ideal $I \subset S$ is bi-homogeneous if it is generated by polynomials which are homogeneous in the variables x_i and also homogeneous in the variables y_j (at the same time). For example the polynomial $x_0^2 x_1 y_0^2 + x_0 x_1^2 y_0 y_1$ is homogeneous (of degree 3) in the x_i and homogeneous (of degree 2) in the y_j.

Consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. Take the bi-homogeneous ideal $I \subset \mathbf{k}[x_0, x_1, y_0, y_1]$ generated by $f = x_0^2 y_0^2 + x_1^2 y_1^2$. Let X be defined as follows:

$$X = \{((x_0 : x_1), (y_0 : y_1)) \mid f(x_0, x_1, y_0, y_1) = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

That is, X is the zero set of the bi-homogeneous ideal I in the product $\mathbb{P}^1 \times \mathbb{P}^1$. Show that the image of X under the Segre map is a closed subvariety of $\mathbb{P}^3$. In other words, show that X is a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ in the usual sense. To do this, you need to show that the image of X in $\mathbb{P}^3$ under the Segre map is the zero set of a homogeneous polynomial (homogeneous in the usual sense). Hint: it is easy! Just checking definitions.

In general, closed subvarieties of $\mathbb{P}^n \times \mathbb{P}^m$ correspond to radical bi-homogeneous ideals in S.

Problem 3: Let

$$f(x, y) = c_1 + c_2x^2 + c_3y$$
$$g(x, y) = d_1 + d_2x^2 + d_3y,$$

where $c_1, c_2, c_3, d_1, d_2, d_3 \in \mathbb{C}$ are "general".

(a) Find the Newton polytopes of $\Delta(f)$ and $\Delta(g)$. Recall that for a Laurent polynomial $f(x) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha$, the Newton polytope is the convex hull of the exponents α, that is:

$$\Delta(f) = \mathrm{conv}\{\alpha \mid c_\alpha \neq 0\} \subset \mathbb{R}^n$$

(b) Use the Bernstein-Khovanskii-Kushnirenko theorem to find the number of solutions $(x, y) \in (\mathbb{C}^*)^2$ of the system $f(x, y) = g(x, y) = 0$.

Problem 4: Let the morphism $\Phi : \mathbb{A}^2 \to \mathbb{P}^3$ be given by

$$\Phi(x, y) = (x^3 : x^2y : xy : 1),$$

and let Y be the closure of the image of $\mathbb{A}^2$ in $\mathbb{P}^3$. Let $\mathbf{k}[Y]$ denote the homogeneous coordinate ring of Y as a subvariety of $\mathbb{P}^3$ and also let $\mathbf{k}[Y]_m$ be the m-th homogeneous piece of the graded algebra $\mathbf{k}[Y]$.

(a) Show that the vector space $\mathbf{k}[Y]_1$ is isomorphic to the vector space of polynomials $\mathrm{span}\langle x^3, x^2y, xy, 1\rangle$. Find a similar isomorphism for $\mathbf{k}[Y]_2$.

(b) Let $H_Y(m) = \dim_{\mathbf{k}} \mathbf{k}[Y]_m$ be the Hilbert function of Y. Find $H_Y(1)$ and $H_Y(2)$.

(c) (Bonus) Let $P_Y(m)$ be the Hilbert polynomial of Y. Find degree and leading coefficient of P_Y. Find degree of Y as a projective subvariety of $\mathbb{P}^3$. Hint: (As we did in class) show that $H_Y(m)$ is equal to the number of points in the set $A + \cdots + A$ (m times) where $A = \{(3, 0), (2, 1), (1, 1), (0, 0)\}$.

Problem 5: Let $C \subset \mathbb{A}^2$ be the curve defined by the irreducible polynomial

$$x^6 + y^6 - xy = 0.$$

(a) Find singular point(s) of C.

(b) Let $\pi : X \to \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at the origin O. Let $\tilde{C}$ be the strict transform of C, i.e. $\tilde{C} = \overline{\pi^{-1}(C \setminus \{O\})}$. Describe the points in $\tilde{C}$ which lie above $O \in C$.