MATH 2810 Algebraic Geometry, Homework 1

Kiumars Kaveh

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- In all the problems $\mathbf{k}$ denotes the ground field and is assumed to be algebraically closed.

Problem 1: Let $X = V(f) \subset \mathbb{A}^n$ be an affine variety defined by a single polynomial f. Prove that $\dim(X) = n - 1$.

Problem 2: Let $X \subset \mathbb{A}^n$ be an algebraic set with $\overline{X}$ its closure in $\mathbb{P}^n$. Show that X is irreducible if and only if $\overline{X}$ is irreducible.

Problem 3: Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring. Recall that an ideal $I \subset A$ is homogeneous if it is generated by homogeneous elements.

(a) Show that if I is homogeneous then the following holds: let $f = \sum_i f_i \in A$ where f_i denotes the homogeneous degree i component of f. Then $f \in I$ if and only if $f_i \in I$, for all i.

(b) Show that if I is homogeneous then its radical $\sqrt{I}$ is also homogeneous.

Problem 4: (Rational normal curve) Consider the n-uple morphism (also called a Veronese map) $\phi : \mathbb{P}^1 \to \mathbb{P}^n$ given by $(t, s) \mapsto (t^n : st^{n-1} : \cdots : s^n)$. Prove that the image of ϕ is a projective subvariety C_n of $\mathbb{P}^n$ (that is, it is a Zariski closed subset of $\mathbb{P}^n$).

Problem 5: Consider the hyperbola $X = V(xy - 1)$ and the parabola $Y = V(x^2 - y)$ as affine varieties in $\mathbb{A}^2$ (the names hyperbola and parabola are motivated by the case $\mathbf{k} = \mathbb{R}$). Let $\mathbb{P}^2$ be the projective plane with coordinates $(x : y : z)$, and $\mathbb{A}^2 = \{(x : y : z) \mid z \neq 0\}$. Let $\overline{X}$ and $\overline{Y}$ denote the closures of X and Y in $\mathbb{P}^2$ respectively.

(a) Find homogeneous ideals defining $\overline{X}$ and $\overline{Y}$ in $\mathbb{P}^2$.

(b) How many points of intersection do $\overline{X}$ and $\overline{Y}$ have?

(c) Find an automorphism ϕ of the projective plane $\mathbb{P}^2$ which maps $\overline{X}$ to $\overline{Y}$.

Problem 6: Let $X = V(y - x^2, z - x^3) \subset \mathbb{A}^3$. Prove the following:

(a) $I = I(X) = \langle y - x^2, z - x^3 \rangle$.

(b) Let $\bar{I}$ be the homogenization of I, that is, ideal in $\mathbf{k}[x, y, z, w]$ generated by the homogenizations of all the elements of I. Show that $zw - xy \in \bar{I}$ but $zw - xy \notin \langle wy - x^2, w^2z - x^3 \rangle$. That is, $\bar{I}$ is not generated by homogenizations of a set of generators for I.

Problem 7: Consider the 3-uple embedding from $\phi : \mathbb{P}^1 \to \mathbb{P}^3$:

$$(s : t) \mapsto (t^3 : t^2s : ts^2 : s^3).$$

Let Y be the image of this map. We know that Y is an irreducible projective algebraic subvariety of $\mathbb{P}^3$ whose homogeneous ideal I in $\mathbf{k}[x, y, z, w]$ is generated by $F_0 = xz - y^2$, $F_1 = yw - z^2$ and $F_2 = xw - yz$. Show that no two of the polynomials F_0, F_1, F_2 can generate I.

Other problems (no need to hand in)

Problem: Let $X \subset \mathbb{P}^n$ be a projective algebraic variety. Suppose $X \cap U_0 \neq \emptyset$ where U_0 is the open chart in $\mathbb{P}^n$ defined by $x_0 \neq 0$. Show that dimensions of X (as a projective subvariety of $\mathbb{P}^n$) and $X \cap U_0$ (as an affine variety in $U_0 = \mathbb{A}^n$) coincide. Use this to give a proof that the rational normal curve has dimension 1.

Problem: Let $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ be defined by $t \mapsto (t^2, t^3)$. Show that ϕ is a morphism which is a homeomorphism onto the curve $y^2 - x^3 = 0$ but ϕ is not an isomorphism.

Problem: Give a continuous map from the projective plane $\mathbb{CP}^2$ onto the (standard) simplex Δ with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ such that the inverse image of each point in the interior of Δ is the topological torus $S^1 \times S^1$ (donut shape).

Problem:

(a) Let $f(x, y)$, $g(x, y)$ be homogeneous polynomials of the same degree d defining a morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $\phi(x : y) = (f(x, y) : g(x, y))$. Prove that ϕ is an automorphism if and only if f, g are linear functions that are linearly independent. In other words, $\phi \in \mathrm{PGL}(2)$.

(b) Write 0, 1, ∞ for the points $(0 : 1)$, $(1 : 1)$ and $(1 : 0)$ on $\mathbb{P}^1$. Show that for any distinct points $P_0, P1, P_2$ on $\mathbb{P}^1$ there is an automorphism ϕ such that:

$$\phi(0) = P_0, \quad \phi(1) = P_1, \quad \phi(\infty) = P_2.$$

Remark: This is not true for four points, that is, automorphism group of $\mathbb{P}^1$ is not 4-transitive. This is related to the notion of *cross ratio*.