MATH 2810 Algebraic Geometry, Homework 2

Kiumars Kaveh

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- In all the problems $\mathbf{k}$ denotes the ground field and is assumed to be algebraically closed. Also an algebraic variety (unless otherwise specified) means a quasi-projective variety, that is a Zariski open subset of a projective variety. Quasi-projective varieties by definition include projective varieties, affine varieties and their Zariski open subsets.

Problem 1: Consider the hyperbola $X = V(xy - 1)$ and the parabola $Y = V(x^2 - y)$ as affine varieties in $\mathbb{A}^2$ (the names hyperbola and parabola are motivated by the case $\mathbf{k} = \mathbb{R}$). Let $\mathbb{P}^2$ be the projective plane with coordinates $(x : y : z)$, and $\mathbb{A}^2 = \{(x : y : z) \mid z \neq 0\}$. Let $\overline{X}$ and $\overline{Y}$ denote the closures of X and Y in $\mathbb{P}^2$ respectively.

(a) Find homogeneous ideals defining $\overline{X}$ and $\overline{Y}$ in $\mathbb{P}^2$.

(b) How many points of intersection do $\overline{X}$ and $\overline{Y}$ have?

(c) Find an automorphism ϕ of the projective plane $\mathbb{P}^2$ which maps $\overline{X}$ to $\overline{Y}$.

Problem 2: Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a radical ideal. Prove that homogenization of I is a radical ideal in $\mathbb{C}[x_0, \ldots, x_n]$.

Problem 3: Show that the projective space $\mathbb{P}^n$ is an abstract variety (see definitions in Appendix A in Karen Smith's, you can skip some straightforward details but please explain clearly.)

Problem 4: Let X be the curve in $\mathbb{A}^2$ defined by the equation $y^2 = x^3 + ax + b$ (it is usually called an *elliptic curve.* Show that the closure $\overline{X}$ of X in $\mathbb{P}^2$ has one extra point which we call the *point at infinity* O.

Show that the line at infinity $\mathbb{P}^2 \setminus \mathbb{A}^2$ is tangent to $\overline{X}$ at O. Hint: let (x_1, x_2) be the coordinates in $\mathbb{A}^2$ and $(x_0 : x_1 : x_2)$ the homogeneous coordinate in $\mathbb{P}^2$. Write down the equation of the curve $\overline{X}$ in the coordinate chart $x_1 \neq 0$.

Problem 5: Show that $\mathbb{C}^2 \setminus \{(0,0)\}$ is a quasi-affine variety (i.e. open subset of an affine variety) but it is not an affine variety (i.e. not isomorphic to an affine variety in some affine space $\mathbb{C}^m$).

Problem 6: Consider the Veronese embedding from $\phi : \mathbb{P}^1 \to \mathbb{P}^3$:

$$(s : t) \mapsto (t^3 : t^2 s : ts^2 : s^3).$$

Let Y be the image of this map. We know that Y is an irreducible projective algebraic subvariety of $\mathbb{P}^3$ whose homogeneous ideal I in $\mathbf{k}[x, y, z, w]$ is generated by $F_0 = xz - y^2$, $F_1 = yw - z^2$ and $F_2 = xw - yz$. Show that no two of the polynomials F_0, F_1, F_2 can generate I.

Problem 7: Consider the variety

$$T = (\mathbf{k}^*)^n = \{(x_1, \ldots, x_n) \mid x_i \neq 0, \ \forall i\},$$

which is an open subset of affine space $\mathbb{A}^n$. The variety T is called the *algebraic torus*. Note that T is a group with multiplication, and when $\mathbf{k} = \mathbb{C}$, T contains the topological torus $(S^1)^n$ as a subgroup.

(a) Show that T is an affine variety. That is, find an isomorphism between T and an affine variety in some affine space $\mathbb{A}^N$.

(b) Show that the ring $\mathcal{O}(T)$ of regular functions on T is the ring of Laurent polynomials $\mathbf{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Problem 8: Consider the Grassmannian variety $\mathrm{Gr}(2,3)$. We regard it as the variety of all the projective lines in $\mathbb{P}^2$. Fix an irreducible conic C in $\mathbb{P}^2$ (i.e. an irreducible curve defined by a homogenous quadratic polynomial). Show that the set of lines in $\mathbb{P}^2$ that fail to intersect C in exactly two (distinct) points is a (closed) subvariety of $\mathrm{Gr}(2,3)$. Hint: Let C be given by a quadratic polynomial $p(x, y) = 0$. Write the condition for a line $ax + by + bz = 0$ to intersect the conic C in one point (note that any line and C have at least one point of intersection).

Some more problems:

Problem: Let $V \subset \mathbb{A}^n$ be an affine variety. Prove that the Zariski closure $\overline{V} \subset \mathbb{P}^n$ coincides with the closure of V in $\mathbb{P}^n$ with respect to the Euclidean topology. Hint: Let $\overline{V}_Z$ and $\overline{V}_E$ denote the closure with respect to the Zariski topology and the Euclidean topology. Since the Zariski topology is coarser that the Euclidean topology we have $\overline{V}_E \subset \overline{V}_Z$. Show the other inclusion.

Problem: Given an integral point $\mathbf{a} = (\alpha_1, \ldots, \alpha_n)$ let $x^{\mathbf{a}}$ denote the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Take a finite subset of integral point $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\} \subset \mathbb{Z}^n$. Consider the morphism of algebraic varieties $\Psi_A : T \to \mathbb{A}^N$ given by

$$\Psi_A : x = (x_1, \ldots, x_n) \mapsto (x^{\mathbf{a}_1}, \ldots, x^{\mathbf{a}_N}).$$

Let X_A denote the closure of the image of Ψ_A in $\mathbb{A}^N$.

(a) Prove that Ψ_A is one-to-one if and only if the set A generates $\mathbb{Z}^n$ as a group. (Hint: note that $\Psi_A : T \to (\mathbf{k}^*)^N$ is a homomorphism of multiplicative groups.)

(b) Prove that the coordinate ring of X_A is the subalgebra of the ring of Laurent polynomials $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by the monomials $\{x^{\mathbf{a}} \mid \mathbf{a} \in A\}$.

Problem: With notation as in the previous problem, show that if the image of Ψ_A is closed in the affine space $\mathbb{A}^N$ then the origin in $\mathbb{R}^n$ should lie in the convex hull $\Delta(A)$ of A. (The converse statement is also true and follows from the famous Hilbert-Mumford criterion in invariant theory.) Hint: show that if the origin is outside $\Delta(A)$ then the origin O in $\mathbb{A}^N$ is in the closure of $\Psi(T)$ and O is not in $\Psi(T)$ by definition of Ψ. To do this, show that there is a curve $\gamma(t) = (t^{e_1}, \ldots, t^{e_N})$, for some non-negative integers e_i, such that for $t \neq 0$, $\gamma(t)$ lies in $\Psi(T)$ and clearly $\lim_{t \to 0} \gamma(t) = O$.

Problem: Equip $\mathbb{P}^n \times \mathbb{P}^m$ with the topology given by the Zariski topology on its image under the Segre map. Show that a subset $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ is closed if and only if it is defined by a collection of polynomials in $\mathbf{k}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ which are homogeneous separately in the variables x_i and in the variables y_j.

Problem: Prove the following:

(a) Let $\mathrm{char}(\mathbf{k}) = p$. Show that the Frobenious morphism $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ given by $x \mapsto x^p$ is a homeomorphism (with respect to Zariski topology) but is not an isomorphism (i.e. ϕ^{-1} is not a morphism).

(b) Let $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ be defined by $t \mapsto (t^2, t^3)$. Show that ϕ is a morphism which is a homeomorphism onto the curve $y^2 - x^3 = 0$ but ϕ is not an isomorphism.