MATH 2810 Algebraic Geometry, Homework 1

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- In all the problems $\mathbf{k}$ denotes the ground field and is assumed to be algebraically closed.

Problem 1:

(a) Show that the Zariski topology on an algebraic set is quasi-compact in the sense that any open cover has a finite subcover.

(b) Show that Zariski topology on an algebraic set is not Hausdorff unless it is a finite set. In fact, show that a Hausdorff space is Noetherian if and only if it is finite.

Problem 2:

(a) Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$. (Hint: Consider the diagonal.)

(b) Show that every non-empty Zariski open set is dense in $\mathbb{A}^n$.

Problem 3: Let X be an algebraic set. Show that there is a one-to-one correspondence between points of X and maximal ideals in $\mathbf{k}[X]$. Moreover, show that there is a one-to-one correspondence between irreducible closed subsets of X and prime ideals in $\mathbf{k}[X]$.

Problem 4: Let $\mathbf{k}$ be a field of characteristic $\neq 2$. Decompose the algebraic set $X \subset \mathbb{A}^3$ defined by the equations $x^2+y^2+z^2 = 0$ and $x^2-y^2-z^2+1 = 0$, into irreducible components.

Problem 5: Let $X = \{(t^2, t^3) \mid t \in \mathbf{k}\}$ be a twisted cubic curve. Show that X is an irreducible algebraic variety which is not isomorphic to the affine line $\mathbb{A}^1$. On the other hand, let $\mathbf{k}(X)$ be the quotient field of the coordinate ring $\mathbf{k}[X]$ (this is called the *field of rational functions on* X). Show that the field $\mathbf{k}(X)$ is isomorphic (as a $\mathbf{k}$-algebra) to the field of rational polynomials $\mathbf{k}(t)$ (in one variable t).

Problem 6: Let $X \subset \mathbb{A}^{n+1}$ be a hypersurface defined by a polynomial $f \in \mathbf{k}[t_1, \ldots, t_n, x]$. Let

$$f = a_m x^m + \cdots + a_0$$

where $a_i \in \mathbf{k}[t_1, \ldots, t_n]$. Consider the projection map $\varphi : X \to \mathbb{A}^n$ given by $(t_1, \ldots, t_n, x) \mapsto (t_1, \ldots, t_n)$. Prove that φ is a finite map if and only if a_m is a nonzero constant polynomial.

Problem 7: Give an example of a morphism $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ whose image is neither closed nor open.

Some more problems (no need to hand in):

Problem: Find the Zariski closure of the graph of the function $y = e^x$ in the affine space $\mathbb{C}^2$. You can use the fact that e^x is not algebraic over the field of rational polynomials $\mathbb{C}(x)$.

Problem: Show that a radical ideal I in the ring $\mathbf{k}[x_1, \ldots, x_n]$ is the intersection of all the maximal ideals containing I.

Problem: Assume that $\mathbf{k}$ is a field of characteristic p. Show that the Frobenius morphism $\mathbb{A}^1 \to \mathbb{A}^1$, $x \mapsto x^p$ is a homeomorphism (that is, a bijection with a continuous inverse). Show that this morphism is not an isomorphism.

Problem: Consider the parameterized curve $C = \{(t^3, t^4, t^5) \mid t \in \mathbf{k}\}$. Is C an irreducible algebraic set in $\mathbb{A}^3$? Prove your claim.

Problem: Consider the algebraic set $V = V(xy + zw) \subset \mathbb{A}^4$.

(a) Show that $xy + zw$ is an irreducible polynomial, and conclude that V is an irreducible variety.

(b) Prove that the coordinate ring $\mathbf{k}[V]$ of V is not a UFD.

Problem: Suppose $\phi : \mathbb{A}^n \to \mathbb{A}^n$ is a morphism. Let $J\phi(x) = det(\partial\phi_i/\partial x_j)$ denote the Jacobian of ϕ at $x = (x_1, \ldots, x_n)$, that is, the determinant of the matrix of partial derivatives. Show that if ϕ is an automorphism, i.e., ϕ is a bijection and ϕ^{-1} is also a morphism, then $J\phi(x)$ is a nonzero constant function. (The famous Jacobian conjecture claims that the opposite is also true.) Give an example of an automorphism of $\mathbb{A}^n$ which is not linear.

Problem: Consider the curve $C = \{(t, t^2, \ldots, t^n) | t \in \mathbf{k}\}$.

(a) Prove that C is an irreducible algebraic set in $\mathbb{A}^n$ of dimension 1. It is usually known as the rational normal curve.

(b) Find generators for the ideal I of C. Show that it can be generated by $n - 1$ elements (i.e. C is a local complete intersection).

(c) Prove that $\mathbf{k}[C] \cong \mathbf{k}[t]$, the polynomial ring in one variable t.

Problem: Show that a variety V in $\mathbb{C}^n$ is bounded (in the usual metric on $\mathbb{C}^n$) if and only if it is a finite set. Use this to show that the set $U(n)$ of $n \times n$ unitary matrices is *not* an algebraic variety of $\mathbb{C}^{n^2} = M(n, \mathbb{C})$, the affine space of all $n \times n$ complex matrices. On the other hand, show that $U(n)$ is an algebraic subvariety of $\mathbb{R}^{2n^2} = M(2n, \mathbb{R})$, the affine space of all $2n \times 2n$ real matrices.