MATH 2810 Algebraic Geometry Homework 1 and some practice problems

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Do problems 1–10.

Throughout **k** denotes an algebraically closed field. In some of the questions the base field is $\mathbb{C}$. You can use material proved in class or in Karen Smith's book.

Problem 1: Show that the zero set of a non-zero polynomial in $\mathbb{C}[x, y]$ does not have interior points (with respect to the usual Euclidean topology on $\mathbb{C}^2$). (The same statement holds with similar proof in all dimensions.)

Problem 2:

(a) Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$. (Hint: Consider the diagonal.)

(b) Show that every non-empty Zariski open set is dense in $\mathbb{A}^n$ (in Zariski topology).

Problem 3: Show that the dimension of an affine algebraic variety is finite.

Problem 4: Show that a radical ideal I in the ring $\mathbf{k}[x_1, \ldots, x_n]$ is the intersection of all the maximal ideals containing I.

Problem 5: Find the Zariski closure of the graph of the function $y = e^x$ in the affine space $\mathbb{C}^2$.

Problem 6: Consider the twisted cubic curve $C = \{(t^3, t^4, t^5) \mid t \in \mathbf{k}\}$. Prove that C is an irreducible algebraic set in $\mathbb{A}^3$ of dimension 1.

Problem 7: Let $\mathbf{k}$ be a field of characteristic $\neq 2$. Decompose the algebraic set $X \subset \mathbb{A}^3$ defined by the equations $x^2+y^2+z^2 = 0$ and $x^2-y^2-z^2+1 = 0$, into irreducible components.

Problem 8: (Related to Gröbner bases and monomial orders) Fix a term order $<$ on $\mathbb{Z}^n$. Let $f(x_1, \ldots, x_n) = \sum_{\alpha=(a_1,\ldots,a_n)} c_\alpha x_1^{a_1} \cdots x_n^{a_n}$ be a non-zero polynomial. Let $v(f)$ denote the highest exponent of f with respect to $<$, that is:

$$v(f) = \max\{\alpha \mid c_\alpha \neq 0\} \in \mathbb{Z}^n_{\geq 0}.$$

Let L be a finite dimensional vector subspace of polynomials. Show that $\dim_{\mathbf{k}}(L)$ is equal to the number of elements in the set $v(L \setminus \{0\}) \subset \mathbb{Z}^n_{\geq 0}$, that is the number of highest exponents appearing in any $f \in L$.

Problem 9: Let $X = \{(t^2, t^3) \mid t \in \mathbf{k}\}$ be a twisted cubic curve. Show that X is an irreducible algebraic variety which is not isomorphic to the affine line $\mathbb{A}^1$. On the other hand, let $\mathbf{k}(X)$ be the quotient field of the coordinate ring $\mathbf{k}[X]$ (this is called the *field of rational functions on* X). Show that the field $\mathbf{k}(X)$ is isomorphic (as a $\mathbf{k}$-algebra) to the field of rational polynomials $\mathbf{k}(t)$ (in one variable t).

Problem 10: Let $X \subset \mathbb{A}^n$ be an algebraic variety. Show that for any $p = (a_1, \ldots, a_n)$ the ideal $\mathfrak{m}_p = \langle x_1 - a_1, \ldots, x_n - a_n\rangle$ is a maximal ideal and consists of all the functions in $\mathbf{k}[X]$ which vanish at p. Show that $p \mapsto \mathfrak{m}_p$ gives a one-to-one correspondence between the points in X and the maximal ideals in $\mathbf{k}[X]$.

Some more problems:

Problem 11: Consider the algebraic set $V = \mathbb{V}(xy + zw) \subset \mathbb{A}^4$.

(a) Show that $xy + zw$ is an irreducible polynomial, and conclude that V is an irreducible variety.

(b) Prove that the coordinate ring $\mathbf{k}(V)$ of V is not a UFD.

Problem 12: Show that in a ring R, a point in Spec$R$ is closed if and only if it corresponds to a maximal ideal. Moreover, show that if R is an integral domain then the prime ideal $\{0\}$ is dense in Spec$R$.

Problem 13: Suppose $\phi : \mathbb{A}^n \to \mathbb{A}^n$ is a morphism. Let $J\phi(x) = det(\partial\phi_i/\partial x_j)$ denote the Jacobian of ϕ at $x = (x_1, \ldots, x_n)$, that is, the determinant of the matrix of partial derivatives. Show that if ϕ is an automorphism, i.e., ϕ is a bijection and ϕ^{-1} is also a morphism, then $J\phi(x)$ is a nonzero constant function. (The famous Jacobian conjecture claims that the opposite is also true.) Give an example of an automorphism of $\mathbb{A}^n$ which is not linear.

Problem 14: Show that in Zariski topology the affine space $\mathbb{A}^n$ is compact (i.e. every open cover has a finite subcover).

Problem 15: Consider the curve $C = \{(t, t^2, \ldots, t^n) | t \in \mathbf{k}\}$.

(a) Prove that C is an irreducible algebraic set in $\mathbb{A}^n$ of dimension 1. It is usually known as the rational normal curve.

(b) Find generators for the ideal I of C. Show that it can be generated by $n-1$ elements (i.e. C is a local complete intersection).

(c) Prove that $\mathbf{k}[C] \cong \mathbf{k}[t]$, the polynomial ring in one variable t.

Problem 16: Show that a variety V in $\mathbb{C}^n$ is bounded (in the usual metric on $\mathbb{C}^n$) if and only if it is a finite set. Use this to show that the set $U(n)$ of $n \times n$ unitary matrices is *not* an algebraic variety of $\mathbb{C}^{n^2} = M(n, \mathbb{C})$, the affine space of all $n \times n$ complex matrices. On the other hand, show that $U(n)$ is an algebraic subvariety of $\mathbb{R}^{2n^2} = M(2n, \mathbb{R})$, the affine space of all $2n \times 2n$ real matrices.

Problem: 17 Let the group S_3 of permutations of three letters act on the polynomial ring $\mathbf{k}[x_1, x_2, x_3]$ by permutation of the variables. Find the ring of invariant polynomials.

Problem 18: Show that if $X \to Y$ is a surjective morphism of affine algebraic varieties, then the dimension of X is at least as large as the dimension of Y.