MATH2810: Introduction to Algebraic Geometry

Take home Final Exam (April 23, 2020)

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Due: by 8am Saturday April 25th.

You only need to do 5 out of 6 problems, although you will receive extra points for doing all of them. In all the problems the ground field $\mathbf{k}$ is an algebraically closed field. You can use theorems from class unless otherwise stated in the problem.

Problem 1: State the following:

(a) Definitions of a ringed space and abstract variety.

(b) Hilbert's theorem on Hilbert polynomial and degree, and the BKK theorem on the number of solutions of a system of Laurent polynomial equations.

Problem 2: Let $q_0, \dots, q_N$ be positive integers with $\gcd(q_0, \dots, q_N) = 1$. Define the *weighted projective space* $\mathbb{P}(q_0, \dots, q_N)$ (as a set) by:

$$\mathbb{P}(q_0, \dots, q_N) = (\mathbb{C}^{N+1} \setminus \{0\}) / \sim,$$

where $\sim$ is the equivalence relation given by:

$$(x_0, \dots, x_N) \sim (y_0, \dots, y_N) \iff \exists 0 \neq \lambda \in \mathbf{k},\ x_i = \lambda^{q_i} y_i,\ \forall i = 0, \dots, N.$$

Similarly to the projective space, we denote the equivalence class of $(x_0, \dots, x_N)$ by $(x_0 : \cdots : x_N)$.

Consider the weighted projective space $\mathbb{P}(1, 1, 2)$ and the map $\Phi : \mathbb{P}(1, 1, 2) \to \mathbb{P}^3$ given by:

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0 x_1 : x_1^2 : x_2).$$

(a) Show that this map is one-to-one and its image $X = \mathrm{Im}(\Phi)$ is a closed subvariety. Find the defining homogeneous equation of $X \subset \mathbb{P}^3$.

(b) Is X a smooth variety? Prove or disprove your claim. Hint: look at X in different affine charts in $\mathbb{P}^3$.

(c) Use the BKK theorem to find the degree of X in $\mathbb{P}^3$. Hint: restrict Φ to an open subset of $\mathbb{P}(1,1,2)$ which is a copy of $(\mathbf{k} \setminus \{0\})^2$.

Problem 3: Analogous to the case of projective space $\mathbb{P}^3$, write the affine charts in the weighted projective space $\mathbb{P}(1,1,2)$ and the gluing maps between them to show that $\mathbb{P}(1,1,2)$ can be realized as an abstract pre-variety.

Problem 4: Find the Hilbert polynomial of $X = \mathbb{P}^n \times \mathbb{P}^m$ embedded in $\mathbb{P}^{(n+1)(m+1)-1}$ via the Serge embedding. Find the degree of X in $\mathbb{P}^{(n+1)(m+1)-1}$.

Problem 5: Consider the affine plane curve $X = V(y^2 - x^3) \subset \mathbb{A}^2$ with coordinate ring $\mathbf{k}[X] = \mathbf{k}[x,y]/(y^2 - x^3)$.

(a) Show that $f = y/x \in k(X)$ is integral over $k[X]$.

(b) Use f to construct the normalization $\tilde{X} \subset \mathbb{A}^3$ and the normalization map $\pi : \tilde{X} \to X$ (verify that $\tilde{X}$ you constructed is actually a normal variety).

Problem 6: Assume we know the following fact: intersection of a projective variety $X \subset \mathbb{P}^N$ with a hyperplane H in general position has dimension $\dim(X) - 1$ (this is a corollary of the well-known Krull's principal ideal theorem or Hauptidealsatz). Prove the following:

(a) Suppose $L \subset \mathbb{P}^N$ is a plane in general position with $\operatorname{codim}(L) > \dim(X)$. Show that $X \cap L$ is empty.

(b) If $U \subset X$ is a nonempty open subset then $\deg(U) = \deg(X)$. Recall that degree is the number of intersection points with a plane of complementary dimension.