MAT2810: Introduction to Algebraic Geometry

Take home Final Exam (April 26, 2015)

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Due: by 11:59 and 59 seconds am Tuesday noon April 28, 2015.

You only need to do the first 4 problems, although you will receive extra points for doing the 5th. In all the problems the ground field $\mathbf{k}$ is an algebraically closed field.

Problem 1:

(a) Give definitions of an affine variety, projective variety and quasi-projective variety.

(b) Give definitions of: (1) a function regular on an open subset U of a variety X, (2) sheaf of regular functions on a variety X (i.e. the structure sheaf $\mathcal{O}_X$), (3) a rational function.

(c) Let $\phi : X \to Y$ be a morphism between (quasi-projective) varieties X and Y. Prove that ϕ is an isomorphism if and only if ϕ is a homeomorphism (for the Zariski topologies) and for each $p \in X$, the induced map $\phi^* : \mathcal{O}_{\phi(p),Y} \to \mathcal{O}_{p,X}$ is an isomorphism of $\mathbf{k}$-algebras.

Problem 2: Let $\mathbf{k} = \mathbb{C}$. Consider the affine curve $X = V(y^2 - x^2(x+1))$ in $\mathbb{A}^2$.

(a) Show that the function y/x (restricted to X) is integral over the coordinate ring $\mathbf{k}[X]$, i.e. it satisfies a monic polynomial with coefficients in $\mathbf{k}[X]$.

(b) Show that y/x is not a regular function on X. More precisely, show that y/x does not coincide (on its domain of definition which is the Zariski open set $\{(x, y) \in X \mid x, y \neq 0\}$) with a polynomial $f(x, y)$ (restricted to X). Hint: suppose $y/x = f(x, y) \pmod{y^2 - x^2(x + 1)}$ and arrive at a contradiction.

Problem 3: This problem is basically a straightforward exercise about line bundles. Recall that a line bundle L on a variety X is given by the information of a (Zariski) open cover $\{U_\alpha\}$ for X and a collection of functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{k}^*$ (for any pair of open sets U_α and U_β) satisfying:

(1) $\forall \alpha, \beta,\ g_{\alpha\beta} = 1/g_{\beta\alpha}$.

(2) $\forall \alpha, \beta, \gamma,\ g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$.

Then L is the disjoint union of the sets $U_\alpha \times \mathbb{A}^1$ modulo the equivalence relation that if $x \in U_\alpha \cap U_\beta$ then we identify the pair $(x, v) \in U_\alpha \times \mathbb{A}^1$ with the pair $(x, g_{\alpha\beta}(x)v) \in U_\beta \times \mathbb{A}^1$. In other words, the (scalar) functions $g_{\alpha\beta}$ are the change of coordinates from one trivializing coordiante chart $U_\alpha \times \mathbb{A}^1$ to another $U_\beta \times \mathbb{A}^1$. One shows that L can be given the structure of an algebraic variety such that for all α the inclusion maps $U_\alpha \times \mathbb{A}^1 \to L$ are morphisms. The maps $(x, v) \mapsto x$ glue together to give the projection map $\pi : L \to X$. For each $x \in X$, the fiber $\pi^{-1}(x)$ is isomorphic to $\mathbb{A}^1$ (but this isomorphism is not canonical).

(a) Recall that a (regular) section σ of L on X is a morphism (regular map) $\sigma : X \to L$ such that $\pi \circ \sigma = \mathrm{id}$. Show that in terms of the data $(\{U_\alpha\}, \{g_{\alpha\beta}\})$, a section σ is given by a collection of regular maps $\sigma_\alpha : U_\alpha \to \mathbb{A}^1$ such that $\sigma_\beta(x) = g_{\alpha\beta}(x)\sigma_\alpha(x)$ for any $x \in U_\alpha \cap U_\beta$.

(b) Let L be the *tautological bundle* on the projective space $\mathbb{P}^n$, i.e.:

$$L = \{(x, v) \mid v \text{ lies on the line representing } x \in \mathbb{P}^n\} \subset \mathbb{P}^n \times \mathbb{A}^{n+1}.$$

Give a trivializing open cover $\{U_\alpha\}$ and change of coordinate functions $\{g_{\alpha\beta}\}$ for L.

(c) Show that the tautological line bundle has no nonzero sections.

(d) Let L^* be the dual line bundle to the tautological line bundle L. it is usually called the *hyperplane bundle*:

$$L^* = \{(x, f) \mid f \text{ is a linear function on the line representing } x\}.$$

The map $\pi : L^* \to \mathbb{P}^n$ is $(x, f) \mapsto x$. The data of L^* is the same as L but with $g_{\alpha\beta}$ replaced with $1/g_{\alpha\beta}$. Show that the space $\Gamma(\mathbb{P}^n, L^*)$ of global sections of L^*, i.e. all the regular sections of L^*, can be identified with the dual vector space $(\mathbb{A}^{n+1})^*$.

Problem 4: Find the Hilbert polynomial of $\mathbb{P}^n$ embedded in $\mathbb{P}^m$ via the d-th Veronese embedding ν_d. Here $m = \binom{n+d}{d} - 1$. Verify that the degree of the Hilbert polynomial is n. Also find the degree of the projective variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^m$.

Problem 5:(Bonus) Consider the affine cubic curve

$$y^2 = x^3 + ax + b,$$

in $\mathbb{A}^2$. We will assume that char $\mathbf{k} \neq 2, 3$. Let X be the projective variety which is the closure of this affine curve in $\mathbb{P}^2$. Let O be the point $(0 : 1 : 0)$ on X. Show that every divisor on X of degree 0 is equivalent to a divisor of the form $P - O$ for some $P \in X$. (Recall that two divisors are equivalent, denoted by $\sim$, if their difference is a principal divisor.) Hint: (1) Let $P, Q, R \in X$ lie on the same line ℓ on X. Show that the divisor $P + Q + R - 3O$ is a principal divisor, i.e. find a rational function $(Ax + By + Cz)/(Dx + Ey + Fz)$ whose divisor is $P + Q + R - 3O$. (2) For $R = (x, y) \in X$ let $R' \in X$ denote the point $(x, -y)$. Use (1) to show that $R + R' - 2O$ is a principal divisor. (We didn't define precisely the order of vanishing of a rational function f at a point on a curve in full generality, although we recall that if $f(x, y)$ is a rational function on the plane and X a smooth curve defined by one polynomial equation $g(x, y) = 0$, the order of vanishing of f at a point a on X is the order of tangency of the zero locus of f and X at a, provided that they intersect, otherwise the oder of tangency is 0.)