

A Constructive Characterization of the Split Closure of a Mixed Integer Linear Program

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Outline

1 Introduction

2 Characterization

3 Lattices

4 Polyhedrality

What is the Split Closure

- Split Cuts:
 - Valid Inequalities “equivalent” to Intersection Cuts, Mixed Integer Gomory Cuts and MIR Cuts.
 - Special case of Balas’s Disjunctive Cuts.
- Closure:
 - Obtained by adding **all** cuts in a class.
 - Class could have infinite number of cuts, so closures are not immediately polyhedrons.
 - Example: Chvátal Closure (Is a polyhedron).

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History and Motivation

- History:
 - Split Cuts were introduced by [Cook, et. al. 1990].
 - Split Closure is a polyhedron [Cook, et. al. 1990, Andersen, et. al. 2005].
Non-constructive proofs.
 - The Split Closure has recently been studied by [Dash et. al. 2008, Balas and Saxena, 2008, Vielma, 2007].
- Motivation of Constructive Characterization:
 - Algorithm to generate Split Closure? (Naive).
 - Helps understand Split Cuts better.
 - For fixed dimension. Is the number of inequalities defining the Split Closure polynomial in the size of the input? (Open even for two inequalities in \mathbb{R}^2).

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Example of a Split Cut

6. Valid Inequalities for Mixed-Integer Sets

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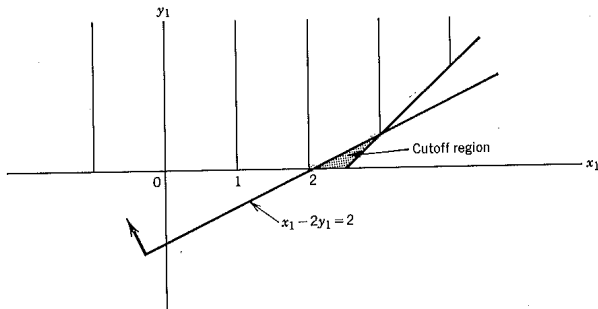


Figure 6.2

Proposition 6.3. *Given the two valid inequalities (6.3) for T , it follows that (6.4) is also valid for T .*

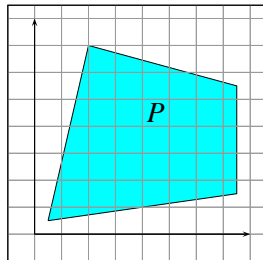
Feasible Set of a (Mixed) Integer Linear Program and Natural Relaxations

Feasible set:

- $P := \{x \in \mathbb{R}^n : a_i^T x \leq b_i \quad \forall i \in M\}$
- $P_I := \{x \in P : x_j \in \mathbb{Z} \quad \forall j \in N_I\}$ for $N_I \subseteq \{1, \dots, n\}$

Relaxations:

- P , LP Relaxation
- $P(B) := \{x \in \mathbb{R}^n : a_i^T x \leq b_i \quad \forall i \in B\}$ for $B \in \mathcal{B} := \{B \subseteq M : |B| = n, \{a_i\}_{i \in B} \text{ l.i.}\}$
Basic or Conic Relaxation
- $\{x \in P(B) : x_j \in \mathbb{Z} \quad \forall j \in N_I\}$ is a relaxation of P_I .
- $x(B)$ unique solution to $a_i^T x = b_i \quad \forall i \in B$



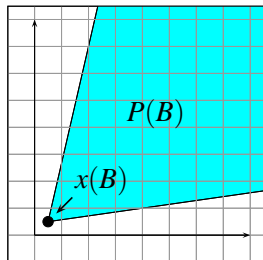
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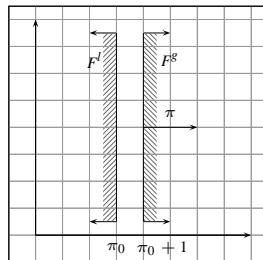
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Split Cuts are Constructed from Valid Split Disjunctions

For $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ divide \mathbb{R}^n into :

- $F^l := \{x \in \mathbb{R}^n : \pi^T x \leq \pi_0\}$
- $F^g := \{x \in \mathbb{R}^n : \pi^T x \geq \pi_0 + 1\}$



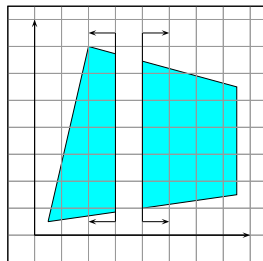
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Use this to divide P into:

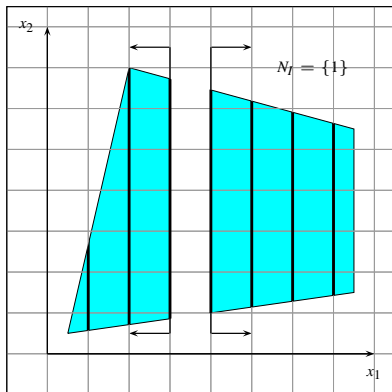
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Valid Splits don't Cut Integer Feasible Points

For fixed N_I we are interested in (π, π_0) such that, for **any** P :

- $P_I \subseteq F^l \cup F^g \subsetneq \mathbb{R}^n$



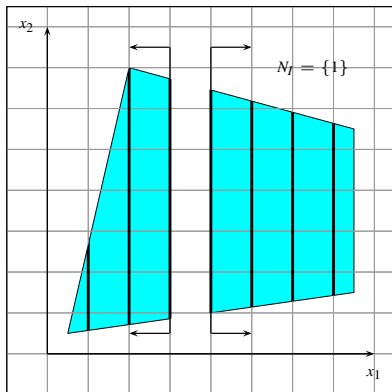
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so we study

- $\Pi(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} : \pi_j = 0, j \notin N_I\}$



The Split Closure is the *Polyhedron* Formed by All Split Cuts

The *split closure* [Cook, et. al. 1990] of P_I is

$$SC := \bigcap_{(\pi, \pi_0) \in \Pi(N_I)} \text{conv}(P_{(\pi, \pi_0)}^l \cup P_{(\pi, \pi_0)}^g).$$

Theorem

[Cook, et. al. 1990] SC is a polyhedron

Sufficient to Study Split Cuts for Basic Relaxations

For basis $B \in \mathcal{B}$ let

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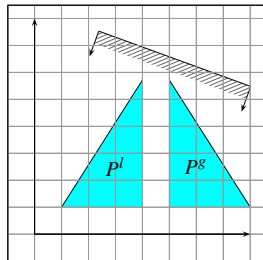
[Andersen, et. al. 2005] $SC = \bigcap_{B \in \mathcal{B}} SC(B)$

Theorem

[Andersen, et. al. 2005] $SC(B)$ is a polyhedron for all $B \in \mathcal{B}$.
Hence SC is a polyhedron.

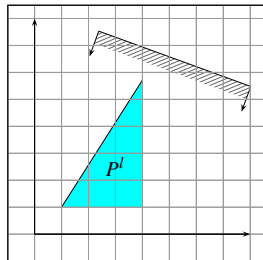
Farkas's Lemma Can be Used to Characterize Split Cuts

- Let $P = P(B) = \{x \in \mathbb{R}^n : Bx \leq b\}$, for $B \in \mathbb{Q}^{n \times n}$, $\text{rank}(B) = n$
- For $(\pi, \pi_0) \in \Pi(N_I)$ such that $\pi^T x(B) \in (\pi_0, \pi_0 + 1)$ let
 - $P^l := \{x \in P : \pi^T x \leq \pi_0\}$
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- Split cut $\delta^T x \leq \delta_0$ is valid for P^l and P^g :
 - F.L. for P^l : $\exists(\mu^l, \mu^l) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ s.t.
 - $\delta = B^T \mu^l + \mu_0^l \pi$
 - $\delta_0 b^T \mu^l + \mu_0^l \pi_0$
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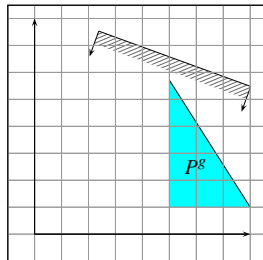
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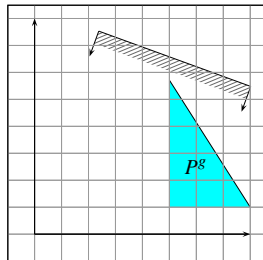
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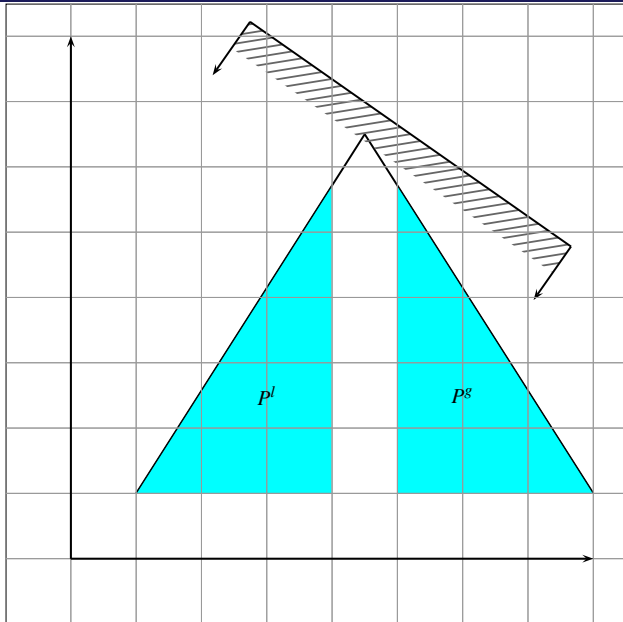
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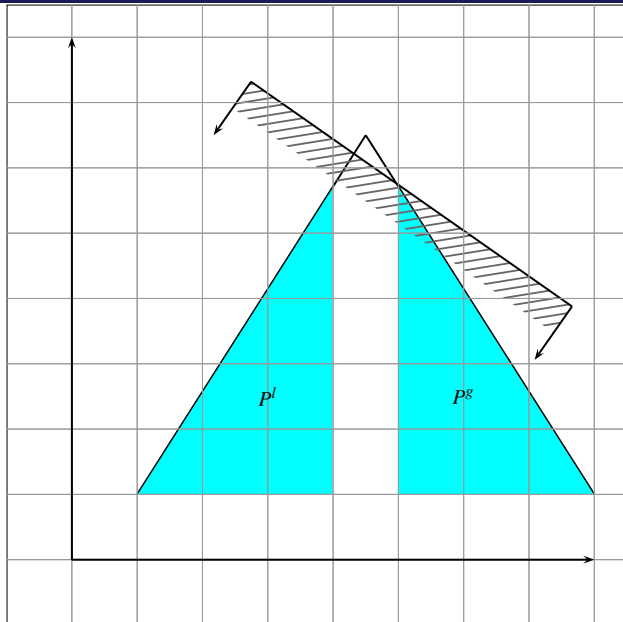


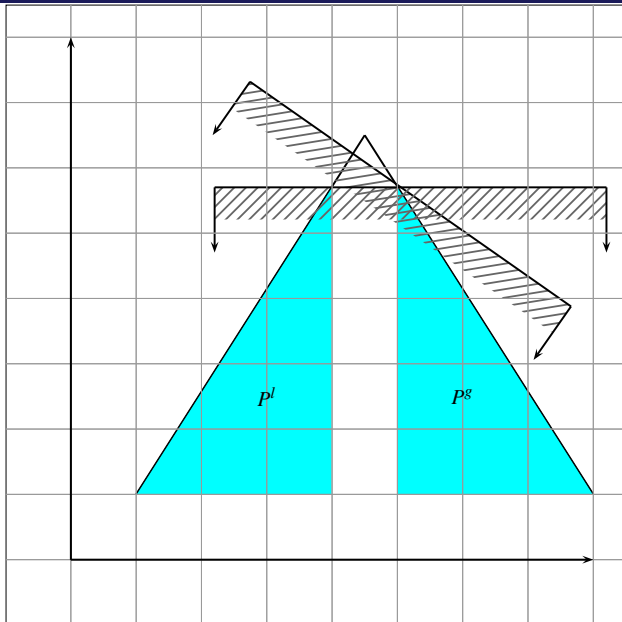
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Proposition

[Andersen, et. al. 2005, Balas and Perregaard, 2003, Caprara and Letchford, 2003] All **non-dominated** valid inequalities for $\text{conv}(P_{(\pi, \pi_0)}^l \cup P_{(\pi, \pi_0)}^g)$ are of the form $\delta^T x \leq \delta_0$ where

$$\delta = B^T \mu^l + \mu_0^l \pi = B^T \mu^g - \mu_0^g \pi$$

$$\delta_0 = b^T \mu^l + \mu_0^l \pi_0 = b^T \mu^g - \mu_0^g (\pi_0 + 1)$$

for $\mu_0^l, \mu_0^g \in \mathbb{R}_+$ and $\mu^l, \mu^g \in \mathbb{R}_+^n$ solutions to

$$B^T \mu^g - B^T \mu^l = \pi$$

$$b^T \mu^g - b^T \mu^l = \pi_0 + \mu_0^g$$

$$\mu_0^l + \mu_0^g = 1, \quad \mu_0^g \in (0, 1), \quad \mu_i^l \cdot \mu_i^g = 0$$

$$B^T \mu^g - B^T \mu^l = \pi$$

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$$\mu^l, \mu^g \in \mathbb{R}_+^n, \quad \mu_i^l \cdot \mu_i^g = 0$$

$$\mu_0^g \in (0, 1), \quad \pi_0 \in \mathbb{Z}$$

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$$B^T \mu = \pi$$

$$b^T \mu = \pi_0 + \mu_0^g$$

$$\mu \in \mathbb{R}^n$$

$$\mu_0^g \in (0, 1), \quad \pi_0 \in \mathbb{Z}$$

$$\mu_i^l = (\mu_i)^- := \max\{-\mu_i, 0\}$$

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$$\mu^T Bx(B) = \mu^T b$$

$$B^T \mu = \pi$$

$$\lfloor b^T \mu \rfloor = \pi_0$$

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$$\pi_0 < \pi^T x(B) < \pi_0 + 1$$

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$$\mu_i^l = (\mu_i)^- := \max\{-\mu_i, 0\}, \quad \mu_0^g = f(b^T \mu), \quad \mu_0^l = 1 - \mu_0^g$$

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Proposition

$$\text{conv}(P_{(\pi, \pi_0)}^l \cup P_{(\pi, \pi_0)}^g) = \{x \in P : \delta^T x \leq \delta_0\}$$

where $\delta(\mu)^T x \leq \delta_0(\mu)$ is

$$(\mu^-)^T (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0$$

for μ unique solution (if it exists) to

$$\begin{aligned} B^T \mu &= \pi & \mu &\in \mathbb{R}^n \\ \mu^T b &\notin \mathbb{Z} & \pi_0 &= \lfloor \mu^T b \rfloor \end{aligned}$$

($y^- = \max\{-y, 0\}$, $f(y) = y - \lfloor y \rfloor$ and operations over vectors are component wise)

What Multipliers Induce Valid Split Disjunctions?

- We have

$\Pi(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} : \pi_j = 0, j \notin N_I\}$ and

$$\begin{aligned} B^T \mu &= \pi & \mu &\in \mathbb{R}^r \\ \mu^T b &\notin \mathbb{Z} & \pi_0 &= \lfloor \mu^T b \rfloor \end{aligned}$$

- Let $B = [B_I B_C]$ for $B_I \in \mathbb{R}^{n \times |N_I|}$ and $B_C \in \mathbb{R}^{n \times (n - |N_I|)}$ corresponding to the integer and continuous variables of P_I . Multipliers that induce valid split disjunctions are

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Valid Split Disjunctions are Related to Integer Lattices

- For $\{v^i\}_{i=1}^r \subseteq \mathbb{R}^n$ l.i. a lattice is

$$\mathcal{L} := \{\mu \in \mathbb{R}^n : \mu = \sum_{i=1}^r k_i v^i \quad k_i \in \mathbb{Z}\}$$

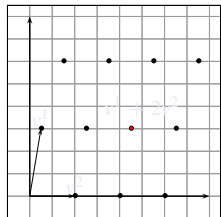
- $\mathcal{L}(B)$ is a lattice,

$$[\mu^-]^T (Bx - b) + (1 - f(\mu^T b)) (\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0$$

is valid for P_I and cuts $x(B)$.

[Köppe and Weismantel, 2004].

- Every $\mu \in \mathcal{L}(B)$ s.t. $\mu^T b \notin \mathbb{Z}$ induces a valid split disjunction.
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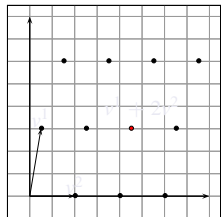
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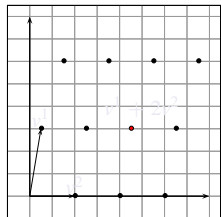
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Proposition

$$SC(B) = \bigcap_{\substack{\mu \in \mathcal{L}(B) \\ \mu^T b \notin \mathbb{Z}}} \{x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu)\}.$$

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For $\mu \in \mathcal{L}(B)$ s.t. $\mu^T b \notin \mathbb{Z}$ split cut

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Studying $\mathcal{L}(B)$ in Each Orthant Decomposes $SC(B)$ to the Intersection of a *Finite* Number of Sets

For $\sigma \in \{0, 1\}^n$ let

$$\mathcal{L}(B, \sigma) := \{\mu \in \mathcal{L}(B) : (-1)^{\sigma_i} \mu_i \geq 0, \quad \forall i \in \{1, \dots, n\}\}$$

so that

$$SC(B) = \bigcap_{\sigma \in \{0, 1\}^n} SC(B, \sigma)$$

where

$$SC(B, \sigma) = \bigcap_{\substack{\mu \in \mathcal{L}(B, \sigma) \\ \mu^T b \notin \mathbb{Z}}} \{x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu)\}$$

Studying $\mathcal{L}(B, \sigma)$ Allows Detecting Dominated Cuts

Lemma

Let $\sigma \in \{0, 1\}^n$ and let $\mu \in \mathcal{L}(B, \sigma)$ with $\mu = \alpha + \beta$ for $\alpha, \beta \in \mathcal{L}(B, \sigma)$ such that $\beta^T b \in \mathbb{Z}$. Then $\delta(\mu)^T x \leq \delta_0(\mu)$ is dominated by $\delta(\alpha)^T x \leq \delta_0(\alpha)$ in $P(B)$.

Proof.

Uses the fact that for α, β in the same orthant $|\alpha_i + \beta_i| = |\alpha_i| + |\beta_i|$ for all $i \in \{1, \dots, n\}$ and the following alternative characterization of split cuts

$$|\bar{\mu}|^T (\bar{B}x - \bar{b}) + (1 - 2f(\bar{\mu}^T \bar{b}))(\bar{\mu}^T \bar{B}x - \lfloor \bar{\mu}^T \bar{b} \rfloor) + f(\bar{\mu}^T \bar{b}) \leq 0$$



A Finite Integral Generating Set (FIGS) of $\mathcal{L}(B, \sigma)$ Induces a Finite Subset of $\mathcal{L}(B, \sigma)$

- Let $\{v^i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathcal{L}(B, \sigma)$ be a (FIGS), i.e. a **finite** set such that

$$\mathcal{L}(B, \sigma) = \left\{ \mu \in \mathbb{R}^r : \mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i \quad k_i \in \mathbb{Z}_+ \right\}$$

- We want $\mu^T b \notin \mathbb{Z}$, so for $i \in \mathcal{V}(\sigma)$ let

$$m_i = \min\{m \in \mathbb{Z}_+ \setminus \{0\} : m b^T v^i \in \mathbb{Z}\}$$

and define the following **finite** subset of $\mathcal{L}(B, \sigma)$.

$$\mathcal{L}^0(B, \sigma) := \left\{ \mu \in \mathcal{L}(B, \sigma) : \mu = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i, r_i \in \{0, \dots, m_i - 1\} \right\}$$

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Proving the Polyhedrality of $SC(B, \sigma)$ Yields the Polyhedrality of SC

Theorem

$SC(B, \sigma)$ the *polyhedron* given by

$$SC(B, \sigma) = \bigcap_{\substack{\mu \in \mathcal{L}^0(B, \sigma) \\ \mu^T b \notin \mathbb{Z}}} \{x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu)\}$$

Corollary

$SC(B)$ is a polyhedron for all $B \in \mathcal{B}$. SC is a polyhedron.

Proof Idea.

- **Goal:** For $\mu \in \mathcal{L}(B, \sigma)$, $\delta(\mu)^T x \leq \delta_0(\mu)$ is dominated by $\delta(\alpha)^T x \leq \delta_0(\alpha)$ for some $\alpha \in \mathcal{L}^0(B, \sigma)$.
- **How:**
 - For $\mu \in \mathcal{L}(B, \sigma)$ show that $\mu = \alpha + \beta$ for α, β such that:
 - $\alpha \in \mathcal{L}^0(B, \sigma)$, $\beta \in \mathcal{L}(B, \sigma)$
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 - Use Lemma.



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Proof of Theorem.

Let $\{v^i\}_{i \in \mathcal{V}(\sigma)}$ be a FIGS for $\mathcal{L}(B, \sigma)$ and let $\{k_i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+$ be such that

$$\mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i.$$



Proof of Theorem.

Let $\{v^i\}_{i \in \mathcal{V}(\sigma)}$ be a FIGS for $\mathcal{L}(B, \sigma)$ and let $\{k_i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+$ be such that

$$\mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i.$$

For each $i \in \mathcal{V}(\sigma)$ we have

$$k_i = n_i m_i + r_i$$

for some $n_i, r_i \in \mathbb{Z}_+$, $0 \leq r_i < m_i$. Let

$$\alpha = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i \quad \text{and} \quad \beta = \sum_{i \in \mathcal{V}(\sigma)} n_i m_i v^i$$

We have $\alpha \in \mathcal{L}^0(B, \sigma)$ and, as m_i is such that $m_i b^T v^i \in \mathbb{Z}$ we have $b^T \beta \in \mathbb{Z}$. □

Final Remarks

- The proof of the Theorem gives a way of enumerating the inequalities of $SC(B, \sigma)$, $SC(B)$ and SC :
 - Not practical for anything but toy problems.
 - There is redundancy in the enumeration for SC and $SC(B)$.
 - There is also redundancy in the enumeration of $SC(B, \sigma)$. In fact we can reduce $\mathcal{L}^0(B, \sigma)$ to

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and $\{r_i\}_{i \in \mathcal{V}(\sigma)}$ are relatively prime

- [Dash et. al. 2008] also give a constructive characterization with similar properties.

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