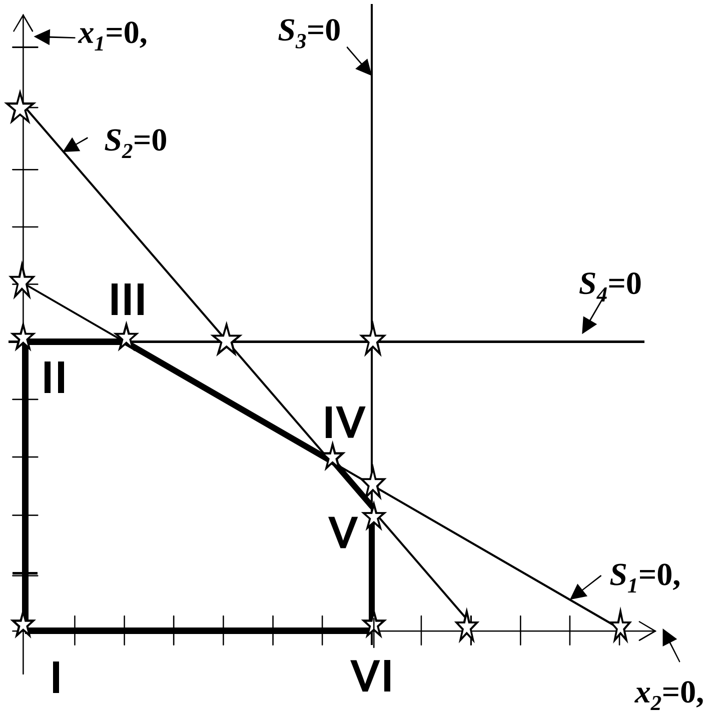
Adjacent Basic Feasible Solutions

	x_1	x_2	S_1	S_2	S_3	S_4
I	0	0	120	90	70	50
II	0	50	20	40	70	0
III	20	50	0	20	50	0
IV	60	30	0	0	10	20
V	70	20	10	0	0	30
VI	70	0	50	20	0	50

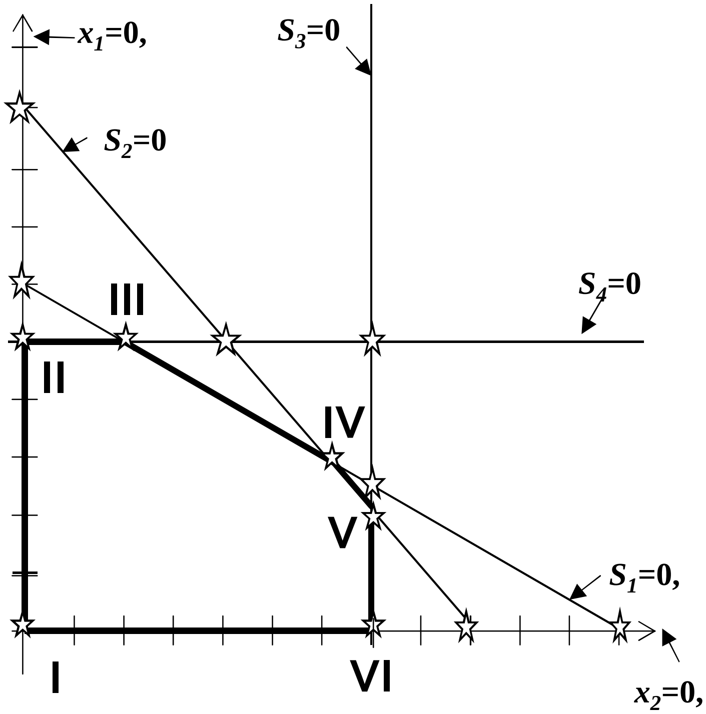


	N	B
I	(x_1, x_2)	(S_1, S_2, S_3, S_4)
II	(x_1, S_4)	(S_1, S_2, S_3, x_2)
II	(x_1, S_4)	(S_1, S_2, S_3, x_2)
III	(S_1, S_4)	(x_1, S_2, S_3, x_2)
III	(S_1, S_4)	(x_1, S_2, S_3, x_2)
IV	(S_1, S_2)	(x_1, S_4, S_3, x_2)
IV	(S_1, S_2)	(x_1, S_4, S_3, x_2)
V	(S_3, S_2)	(x_1, S_4, S_1, x_2)
V	(S_3, S_2)	(x_1, S_4, S_1, x_2)
VI	(S_3, x_2)	(x_1, S_4, S_1, S_2)
VI	(S_3, x_2)	(x_1, S_4, S_1, S_2)
I	(x_1, x_2)	(S_3, S_4, S_1, S_2)

Adjacent Basic Feasible Solutions

	x_1	x_2	S_1	S_2	S_3	S_4
I	0	0	▭	▭	▭	▭
II	0	▭	▭	▭	▭	0
III	▭	▭	0	▭	▭	0
IV	▭	▭	0	0	▭	▭
V	▭	▭	▭	0	0	▭
VI	▭	0	▭	▭	0	▭

▭ =BASIC



	N	B
I II	(x_1, x_2) (x_1, S_4)	(S_1, S_2, S_3, S_4) (S_1, S_2, S_3, x_2)
II III	(x_1, S_4) (S_1, S_4)	(S_1, S_2, S_3, x_2) (x_1, S_2, S_3, x_2)
III IV	(S_1, S_4) (S_1, S_2)	(x_1, S_2, S_3, x_2) (x_1, S_4, S_3, x_2)
IV V	(S_1, S_2) (S_3, S_2)	(x_1, S_4, S_3, x_2) (x_1, S_4, S_1, x_2)
V VI	(S_3, S_2) (S_3, x_2)	(x_1, S_4, S_1, x_2) (x_1, S_4, S_1, S_2)
VI I	(S_3, x_2) (x_1, x_2)	(x_1, S_4, S_1, S_2) (S_3, S_4, S_1, S_2)

Algebraic Specification of the Simplex Method

Moving to an adjacent BFS is the same as exchanging an element of *B* with an element of *N*, i.e., exchanging a basic variable for a nonbasic variable

PHASE I

STEP 0 (INITIALIZATION): Find an initial basic feasible solution (*BFS*), i.e., an extreme point of the feasible region. If one *cannot* be found the problem is *infeasible:* STOP.

PHASE II

STEP 1 (STOPPING CRITERIA CHECK): Is unboundedness detected? If so, there is no optimum solution: STOP. If not, is there an adjacent extreme point where the objective function is better than at the current one?

That is, can the objective be improved by exchanging one of the currently basic variables for one of the currently nonbasic variables? If not, the current *BFS* is *optimal.* STOP.

Proceed to Step 2

STEP 2 (ITERATIVE STEP): Move to the (better) adjacent extreme point identified above in Step 1 by exchanging a basic variable for a nonbasic one. Then return to Step 1.

The Initial Simplex Tableau

Max $Z=20x_1+10x_2$
st
$x_1+2x_2 + S_1 = 120$
$x_1+ x_2 + S_2 = 90$
$x_1 + S_3 = 70$
$x_2 \quad S_4 = 50$
$x_1, x_2 \geq 0$

Maximize Z

Reduced Costs (or *Relative Profits*)

Row	Z	x_1	x_2	S_1	S_2	S_3	S_4	RHS	Basic
0	1	-20	-10	0	0	0	0	0	Z
1	0	1	2	1	0	0	0	120	S_1
2	0	1	1	0	1	0	0	90	S_2
3	0	1	0	0	0	1	0	70	S_3
4	0	0	1	0	0	0	1	50	S_4

Substitution rates

The system of equations represented above is said to be in ***CANONICAL FORM***:

- Each equation has an "isolated" variable that appears in **only** that equation and has a coefficient of **+1**
- The RHS for each constraint equation is a nonnegative constant

We can rewrite the equations as below:

$$\begin{array}{rl} Z = 0 & + 20x_1 + 10x_2 \\ S_1 = 120 & - x_1 - 2x_2 \\ S_2 = 90 & - x_1 - x_2 \\ S_3 = 70 & - x_1 \\ S_4 = 50 & \quad - x_2 \end{array}$$

S_1, S_2, S_3 and S_4 are **BASIC** variables

x_1 and x_2 (the **NONBASIC** variables) are thus parameters here

NOTE that if these parameters (nonbasic variables) are set to zero the system has essentially been "solved" for Z, S_1, S_2, S_3 and S_4!!

Moving to a Better BFS

Letting the nonbasic variables equal 0, we obtain the **Basic Feasible Solution** $x_1 = x_2 = 0$; $S_1=120$, $S_2=90$, $S_3=70$ and $S_4=50$.

Obviously, this **BFS** is not optimal: from $Z = 0 + 20x_1 + 10x_2$ it is clear that increasing either of x_1 or x_2 will increase Z.

Let us (arbitrarily) select x_1 for increase while maintaining the other basic variables (x_2 in this case) at 0. Each unit increase in x_1 increases Z by 20 units. However as x_1 is increased, the values of the (current) basic variables S_1, S_2, S_3 and S_4 change:

ONE unit increase in x_1 ⇒

1 unit **de**crease in S_1 $(= 120 - x_1 - 2x_2)$

1 unit **de**crease in S_2 $(= 90 - x_1 - x_2)$

1 unit **de**crease in S_3 $(= 70 - x_1)$

0 unit change in S_4 $(= 50 - x_2)$

substitution rates!

Question: HOW MUCH MAY x_1 INCREASE ?

Answer: A further increase in x_1 is "blocked" when one of the basic variables reaches its lower bound (zero). To continue increasing x_1 would cause the non-negativity restriction on this basic variable to be violated! Here we thus have:

- S_1 reaches 0 when x_1 reaches 120/1 = 120 (from $S_1 = 120 - x_1$)
- S_2 reaches 0 when x_1 reaches 90/1 = 90 (from $S_2 = 90 - x_1$)
- S_3 reaches 0 when x_1 reaches 70/1 = 70 (from $S_3 = 70 - x_1$)
- S_4 is unaffected by increases in x_1 (50/0= ∞) (from $S_4 = 70$)

As we increase x_1 the first "block" occurs at ***Minimum*** **{120, 90, 70,∞}** = 70, at which point S_3 goes to 0

Iterating in the Simplex Method

In summary we do the *RATIO TEST* of the form

(current basic variable) ÷ (positive substitution rate)

and pick the basic variable corresponding to the row that yields the MINIMUM RATIO

Next, we would like to "re-solve" the system so that we obtain a canonical form with x_1 being a basic variable and S_3 being nonbasic (and hence equal to zero).

__PIVOT OPERATION__: A sequence of **elementary row operations** which reduce the tableau to canonical form. Consider the current tableau:

Pivot Column

Row	Z	x_1	x_2	S_1	S_2	S_3	S_4	RHS	Basic
0	1	-20	-10	0	0	0	0	0	Z
1	0	1	2	1	0	0	0	120	S_1
2	0	1	1	0	1	0	0	90	S_2
3	0	**1**	0	0	0	1	0	70	S_3
4	0	0	1	0	0	0	1	50	S_4

Pivot Row

Pivot Element

We will pivot on the element in

- The ***column*** corresponding to the variable ***entering*** the basis,
- The ***row*** corresponding to the variable ***leaving*** the basis.

Iterating in the Simplex Method

The pivot column must end up with a **1** in the pivot element's spot and zeros elsewhere; so we

- Add 20*(Row 3) to Row 0
- Add -1*(Row 3) to Row 1
- Add -1*(Row 3) to Row 2

The resulting tableau is

Row	Z	x_1	x_2	S_1	S_2	S_3	S_4	RHS	Basic
0	1	0	-10	0	0	20	0	1400	Z
1	0	0	2	1	0	-1	0	50	S_1
2	0	0	1	0	1	-1	0	20	S_2
3	0	1	0	0	0	1	0	70	x_1
4	0	0	1	0	0	0	1	50	S_4

which represent the **BFS** $S_3=x_2=0$; $S_1=50$, $S_2=20$, $x_1=70$ and $S_4=50$ with $Z=1400$.

ARE WE DONE ?
Again, rewriting Z in terms of the basic variables:

$$Z - 10x_2 + 20S_3 = 1400 \quad \Rightarrow \quad Z = 1400 + 10x_2 - 20S_3$$

Z can be increased from its current value (1400) by increasing x_2

Consider the column of substitution rates for x_2. A unit increase in x_2 will force us to (in order to maintain feasibility)

decrease S_1 by 2 units (from $S_1 = 50 - 2x_2$)
decrease S_2 by 1 unit (from $S_2 = 20 - x_2$)
decrease S_4 by 1 unit (from $S_4 = 50 - x_2$)

Conducting the minimum ratio test, the maximum increase possible in x_2 is given by minimum of $\{50/2, 20/1, \infty, 50/1\} = 20$ at which point x_2 goes to zero

Iterating in the Simplex Method

At this point S_2 goes to zero and leaves the basis. Thus the new basis will have x_2 replacing S_2 in the basis. Now pivot again:

CURRENT TABLEAU

Row	Z	x_1	x_2	S_1	S_2	S_3	S_4	RHS	Basic
0	1	0	-10	0	0	20	0	1400	Z
1	0	0	2	1	0	-1	0	50	S_1
2	0	0	1	0	1	-1	0	20	S_2
3	0	1	0	0	0	1	0	70	x_1
4	0	0	1	0	0	0	1	50	S_4

- Row 0 ← Row 0 +(10)*Row 2
- Row 1 ← Row 1 +(-2)*Row 2
- Row 4 ← Row 4 +(-1)*Row 2

OPTIMAL!

New Tableau

Row	Z	x_1	x_2	S_1	S_2	S_3	S_4	RHS	Basic
0	1	0	0	0	10	10	0	1600	Z
1	0	0	0	1	-2	1	0	10	S_1
2	0	0	1	0	1	-1	0	20	x_2
3	0	1	0	0	0	1	0	70	x_1
4	0	0	0	0	-1	1	1	30	S_4

The Simplex Method Another Example…

Maximize $z = 40x_1 + 60x_2 + 50x_3$

st $10x_1 + 4x_2 + 2x_3 \le 950$, i.e., $10x_1 + 4x_2 + 2x_3 + S_1 = 950$

$2x_1 + 2x_2 \le 410$ $\quad 2x_1 + 2x_2 + S_2 = 410$

$x_1 + 2x_3 \le 610$ $\quad x_1 + 2x_3 + S_3 = 610$

$x_1, x_2, x_3 \ge 0.$

CURRENT TABLEAU **Iteration 0**

Row	Basic	Z	x_1	x_2	x_3	S_1	S_2	S_3	RHS	
0	Z	1	-40	-60	-50	0	0	0	0	
1	S_1	0	10	4	2	1	0	0	950	950/4=237.5
2	S_2	0	2	②	0	0	1	0	410	410/2=205
3	S_3	0	1	0	2	0	0	1	610	∞

Row 0 ← Row 0 +(30)*Row 2; Row 1 ← Row 1 +(-2)*Row 2;
Row 2 ← (1/2)*Row 2

Iteration 1

Row	Basic	Z	x_1	x_2	x_3	S_1	S_2	S_3	RHS	
0	Z	1	20	0	-50	0	30	0	12,300	
1	S_1	0	6	0	②	1	-2	0	130	130/2=65
2	x_2	0	1	1	0	0	½	0	205	∞
3	S_3	0	1	0	2	0	0	1	610	610/2=305

Row *0* ← Row *0* + 25*(Row *1*); Row *3* ← Row *3* + (-1)*(Row *1*);
Row *1* ← (1/2)*(Row *1*)

The Simplex Method Another Example…

Iteration 2

Row	Basic	Z	x_1	x_2	x_3	S_1	S_2	S_3	RHS	
0	Z	1	170	0	0	25	-20	0	15,550	
1	x_3	0	3	0	1	½	-1	0	65	∞
2	x_2	0	1	1	0	0	½	0	205	205/0.5=410
3	S_3	0	-5	0	0	-1	2	1	480	480/2=240

Row 0 ← Row 0 +(10)*Row 3; Row 1 ← Row 1 +(1/2)*Row 3;

Row 2 ← Row 2+(-1/4)*Row 3; Row 3← (1/2)*Row 3

Iteration 3

Row	Basic	Z	x_1	x_2	x_3	S_1	S_2	S_3	RHS
0	Z	1	120	0	0	15	0	10	20,350
1	x_3	0	½	0	1	0	0	½	305
2	x_2	0	9/4	1	0	¼	0	-¼	85
3	S_2	0	-5/2	0	0	-½	1	½	240

OPTIMAL! All nonbasic variables have coefficients in Eq. 0 that are nonnegative. Therefore no neighboring (adjacent) extreme point could be any better.

Some Observations…

Row	Basic	Z	x_1	x_2	x_3	S_1	S_2	S_3	RHS	
0	Z	1	170	0	0	25	-20	0	15,550	
1	x_3	0	3	0	1	½	-1	0	65	∞
2	x_2	0	1	1	0	0	½	0	205	205/0.5=410
3	S_3	0	-5	0	0	-1	2	1	480	480/2=240

Reduced Costs

- For basic variables: always equal to 0.
- For nonbasic variables: could be any value. It is
  - the increase (if negative) or decrease (if positive) in Z for a 1 unit **increase** in that nonbasic variable while all other nonbasic variables remain zero. At the optimum: no negative (positive) reduced costs if maximizing (minimizing)

Objective Values (Z)

- Could be any sign depending on the objective coefficients

Substitution Rates (for nonbasic variables)

- Could be any sign: For a 1 unit increase in a nonbasic variable the rate in a row under the column for that nonbasic variable represents *the decrease* (*if positive*) *or the increase* (*if negative*) *required in the value of the basic variable corresponding to that row, so as to maintain feasibility.*

RHS Values

- Cannot be negative

Minimum Ratio

- Maximum allowable increase in the nonbasic variable chosen to enter, before a basic variable decreases to a value of 0 (and hence becomes nonbasic).

Alternative Optima

Consider Max $Z = 2x_1 + 4x_2$
st $x_1+2x_2 \leq 5$; $x_1+x_2 \leq 4$; $x_1, x_2 \geq 0$

Row	Basic	Z	x_1	x_2	S_1	S_2	RHS
0	Z	1	-2	-4	0	0	0
1	S_1	0	1	2	1	0	5
2	S_2	0	1	1	0	1	4

Entering x_2 and removing S_1 yields

Row	Basic	Z	x_1	x_2	S_1	S_2	RHS
0	Z	1	0	0	2	0	10
1	x_2	0	0.5	1	0.5	0	2.5
2	S_2	0	0.5	0	-0.5	1	1.5

OPTIMAL! But…
we can still enter the NBV x_1 into the basis if we wish

Row	Basic	Z	x_1	x_2	S_1	S_2	RHS
0	Z	1	0	0	2	0	10
1	x_2	0	0	1	1	-1	1
2	x_1	0	1	0	-1	2	3

If a tableau indicates optimality (all reduced cost ≥0 for Max or ≤0 for a Min), but a nonbasic variable has a zero reduced cost and can enter the basis, then we have alternative optima.

Unbounded Objective

Suppose we have chosen an entering variable, i.e., a nonbasic variable with negative reduced cost (maximization) or a positive reduced cost (minimization). However, no leaving variable can be found because all of the substitution rates in the pivot column are either zero or less than zero. It is impossible to conduct the ratio test!

Consider Min $Z = 2x_1 - 6x_2$
st $-x_1 + x_2 \leq 1$; $x_1 - 2x_2 \leq 2$; $x_1, x_2 \geq 0$

Row	Basic	Z	x_1	x_2	S_1	S_2	RHS
0	Z	1	-2	6	0	0	0
1	S_1	0	-1	1	1	0	1
2	S_2	0	1	-2	0	1	2

Entering x_1 and removing S_1 yields

Row	Basic	Z	x_1	x_2	S_1	S_2	RHS
0	Z	1	4	0	-6	0	-6
1	x_2	0	-1	1	1	0	1
2	S_2	0	-1	0	2	1	4

Note that x_1 can enter but no ratio test is possible – that means x_1 can be raised indefinitely (and Z improved by 4 units per unit of increase in x_1) without ever endangering feasibility!

The objective for the problem is thus unbounded.

Breaking Ties

- Tie for the entering variable, i.e., there is a tie for the variable that has the "most negative" (for maximization) or "most positive" (for minimization) reduced cost (value in Eq. 0)
- Tie for the leaving variable, i.e., two or more rows tie for the value of the minimum ratio

In either case, break ties arbitrarily!

However, when the tie is for the leaving variable – the variable in the row that is NOT chosen will be basic at the next iteration but with a value of 0! Why?

As the entering (nonbasic) variable is raised in value, when it hits the value of the minimum ratio, two or more variables that are currently basic **simultaneously** reach zero when their values are adjusted to maintain feasibility. However, to go to an adjacent BFS we can only replace **one** of them in the basis – so the other ones remain in the basis but at a value of zero.

E.g.

Row	Basic	Z	x_1	x_2	S_1	S_2	RHS	
0	Z	1	-1	-2	0	0	0	
1	S_1	0	2	1	1	0	20	20/1=20
2	S_2	0	1	2	0	1	40	40/2=20

Tie for leaving variable

If we pick x_2 to enter there is a tie for the leaving variable. Suppose we break this arbitrarily and pick S_1 to leave.
The next tableau will be obtained by performing the *ero*'s
Row 0 ← Row 0 + 2*Row 1, Row 2 ← Row 2 - 2*Row 1

Row	Basic	Z	x_1	x_2	S_1	S_2	RHS
0	Z	1	3	0	2	0	40
1	x_2	0	2	1	1	0	20
2	S_2	0	-3	0	-2	1	0

Notice that the basic variable that was NOT picked, i.e., S_2, is equal to 0 (but we still did improve by 20*2 = 40 units).

Conversely, suppose we break the tie by picking S_2 to leave. Then the next tableau will be
(after Row 0 ← Row 0 + Row 3, Row 2 ← Row 2 – 0.5*Row 3,
Row 3 ← 0.5*Row 3)

Row	Basic	Z	X_1	x_2	S_1	S_2	RHS
0	Z	1	0	0	0	1	40
1	S_1	0	1.5	0	1	-0.5	0
2	x_2	0	0.5	1	0	0.5	20

Again, notice that the basic variable that was NOT picked, i.e., *S*1 is equal to 0 (but we again improved by 20*2 = 40 units).
These two are examples of **degenerate** basic feasible solutions

DEGENERACY

Consider the following LP:

$$\begin{aligned} &\text{Max } Z = 2X_1 + 3X_2 \\ &st \quad X_1 + X_2 \leq 3 \\ &\quad\quad X_1 + 2X_2 \leq 4 \\ &\quad\quad 4X_1 + 3X_2 \leq 12 \\ &\quad\quad X_1, X_2 \geq 0 \end{aligned}$$

Iteration 0

Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic	
0	1	-2	-3	0	0	0	0	Z	
1	0	1	1	1	0	0	3	S_1	*3/1 =3*
2	0	1	**2**	0	1	0	4	S_2	*4/2 =2*
3	0	4	3	0	0	1	12	S_3	*12/3 =4*

Iteration 1

Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic	
0	1	-0.5	0	0	1.5	0	6	Z	
1	0	0.5	0	1	-0.5	0	1	S_1	*1/0.5 =2*
2	0	0.5	**1**	0	0.5	0	2	X_2	*2/0.5 =4*
3	0	2.5	0	0	-1.5	1	6	S_3	*6/2.5 =2.4*

DEGENERACY

Iteration 2

Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic
0	1	0	0	1	1	0	7	Z
1	0	1	0	2	-1	0	2	X_1
2	0	0	1	-1	1	0	1	X_2
3	0	0	0	-5	1	1	1	S_3

OPTIMAL SOLUTION: All reduced costs are nonnegative and so no further increase is possible in the objective (*Z*).

DEGENERACY

Suppose instead that we had started by bringing X_1 into the basis at the first iteration (rather than X_2):

Iteration 0

Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic	
0	1	-2	-3	0	0	0	0	Z	
1	0	①	1	1	0	0	3	S_1	*3/1 =3*
2	0	1	2	0	1	0	4	S_2	*4/1 =4*
3	0	4	3	0	0	1	12	S_3	*12/4 =3*

Iteration 1

Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic	
0	1	0	-1	2	0	0	6	Z	
1	0	1	1	1	0	0	3	X_1	*3/1 =3*
2	0	0	①	-1	1	0	1	S_2	*1/1 =1*
3	0	0	-1	-4	0	1	0	S_3	∞

DEGENERACY

Iteration 2

Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic
0	1	0	0	1	1	0	7	Z
1	0	1	0	2	-1	0	2	X_1
2	0	0	1	-1	1	0	1	X_2
3	0	0	0	-5	1	1	1	S_3

Same optimal solution as before, but different route…

Recall that at Iteration 1 we had a tie for the leaving variable between S_1 and S_3, and we picked S_1 to leave. Consider what happens if we had broken the tie in favor of S_3.

DEGENERACY

Iteration 0

Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic	
0	1	-2	-3	0	0	0	0	Z	
1	0	1	1	1	0	0	3	S_1	*3/1 =3*
2	0	1	2	0	1	0	4	S_2	*4/1 =4*
3	**0**	**4**	3	0	0	1	12	S_3	*12/4 =3*

Iteration 1

Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic	
0	1	0	-1.5	0	0	0.5	6	Z	
1	0	0	**0.25**	1	0	-0.25	0	S_1	*0/0.25 =0*
2	0	0	1.25	0	1	-0.25	1	S_2	*1/1.25 =0.8*
3	0	1	0.75	0	0	0.25	3	X_1	*3/0.75=4*

DEGENERACY

Iteration 2

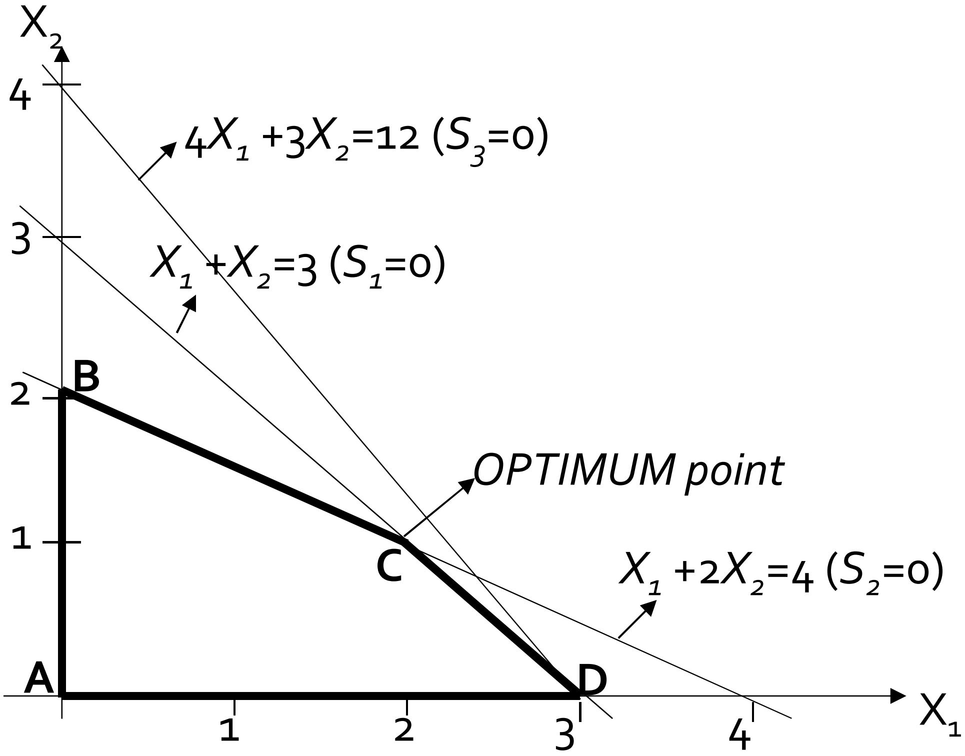
Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic	
0	1	0	0	6	0	-1	6	Z	
1	0	0	1	4	0	-1	0	X_2	∞
2	0	0	0	-5	1	1	1	S_2	$1/1 = 1$
3	0	1	0	-3	0	1	3	X_1	$3/1=3$

Iteration 3

Row	Z	X_1	X_2	S_1	S_2	S_3	RHS	Basic
0	1	0	0	1	1	0	7	Z
1	0	1	0	2	-1	0	2	X_1
2	0	0	1	-1	1	0	1	X_2
3	0	0	0	-5	1	1	1	S_3

Once again, we got to the same optimal solution.
BUT, we took one extra iteration: note that we got temporarily "stuck" at the second iteration - there was no improvement in *Z;* it stayed at 6!

DEGENERACY



Extr. Pt.	BFS No.	Basic Variables (or BASIS)	Nonbasic Variables
A	1	$S_1=3,\ S_2=4,\ S_3=12$	$X_1 = X_2 = 0$
B	2	$S_1=1,\ X_2=2,\ S_3= 6$	$X_1 = S_2 = 0$
C	3	$X_1=2,\ X_2=1,\ S_3= 1$	$S_1 = S_2 = 0$
D	**4**	$X_1=3,\ S_2=1,\ S_3= 0$	$S_1 = X_2 = 0$
D	**5**	$X_1=3,\ S_2=1,\ S_1= 0$	$S_3 = X_2 = 0$
D	**6**	$X_1=3,\ S_2=1,\ X_2= 0$	$S_1 = S_3 = 0$

We took different routes to reach the optimum at C (≈BFS No. 3):

1. BFS1→BFS2→BFS3 (A→B→C)
2. BFS1→BFS4→BFS3 (A→D→C)
3. BFS1→ BFS5→ BFS6→BFS3 (A→D → D→C)

Theorem:

- For every **BFS** with an associated **basis** there is an extreme point that is unique
- For every extreme point there is a corresponding **BFS** with an associated **basis** (that is not *necessarily* unique)
- If there is more than one basis associated with an extreme point, it is said to be degenerate and has more than *n* constraints being active there