I.E. 2001 OPERATIONS RESEARCH

(Solutions to Assignment 5)

Question 1

a) Here x_1, x_4, x_5 are all nonbasic and all have a value of zero, while x_3, x_2, x_6 are basic, with values of 65, 205 and 480 respectively.

b) For minimization x_1 and x_4 are legitimate candidates to enter the basis because both have positive reduced costs (of 170 and 25 respectively), while for maximization x_5 is a legitimate candidate to enter the basis because it has a negative reduced cost (=-20)

c) Since the reduced cost of x_4 is 25, this means that each 1 unit increase in x_4 (while maintaining the other nonbasic variables at their current values of zero) will **de**crease the objective by 25 units. So if we increase x_4 by 100 units the new objective value will decrease by 100*25 = 2500 units, to a new value of 13,050.

d) The substitution rates for the basic variables (from the column fo x_4) are ½ for x_3 (the first basic variable), 0 for x_2 (the second basic variable), and -1 for x_6 (the third basic variable). This means that for a 10 unit increase in x_4:
   - the value of x_3 should decrease by ½(100) =50 units,
   - the value of x_2 should decrease by 0(100) =0 units, and
   - the value of x_6 should decrease by -1(100) units, i.e., **in**crease by 1(100) = 100 units.

e) The maximum actual increase possible in x_4=Min{65/(½), ∞, ∞}=130 units (at which point x_3 will have decreased by ½(130)=65 units from its current value of 65 and reached a value of zero, so that it can now be removed from the basis).

f) Since we can increase x_4 by 130 units (part e) and each unit increase in x_4 reduces the objective by 25 units (part b), the net decrease in Z will be 130×25 = 3,250 units. So the new value will 15,550 – (130×25) = 12,300. (Note also that the new value of x_2 will be 205–(130×0) = 205, and the new value of x_6 will be 480–(130×-1) = 610; and of course x_4 will replace x_3 as a basic variable with a value of 130).

Question 2 (*Q4, p. 213*)

Basic	Z	X_1	X_2	S_1	S_2	RHS	
Z	1	-5	1	0	0	0	
S_1	0	**1**	-3	1	0	1	*1/1*
S_2	0	1	-4	0	1	3	*3/1*

X_1 enters; S_1 leaves

Eq. 0 ← (Eq. 0) + 5*(Eq. 1); Eq. 2 ← (Eq. 2) – (Eq. 1)

Basic	Z	X_1	X_2	S_1	S_2	RHS
Z	1	0	-14	5	0	5
X_1	0	1	-3	1	0	1
S_2	0	0	-1	-1	1	2

At this point, note that X_2 may enter the basis and be increased from its current value of 0 to improve the objective. However, there is *no limit* to the amount of increase in X_2 and the improvement in Z, since neither X_1 nor S_2 need to be decreased to compensate (they both have negative substitution rates and the ratio test cannot be taken). Thus the problem has an unbounded objective and we STOP.

Question 3

First, we put this into standard form

$$\text{Min } Z = 30X_1 + 20X_2$$

$$\begin{aligned} st \quad X_1 \quad & -S_1 = 4 \\ 2X_1 + X_2 \quad & = 20 \\ X_1 + 2X_2 \quad & -S_3 = 19 \\ X_1, X_2, S_1, S_3 & \geq 0. \end{aligned}$$

Since there is no isolated variable in any constraint we need artificial variables for each of them. After adding artificial variables we get the following Phase 1 problem:

<u>Phase 1</u>

$$\text{Min } W = A_1 + A_2 + A_3$$

$$\begin{aligned} st \quad X_1 \quad -S_1 + A_1 \quad & = 4 \\ 2X_1 + X_2 \quad + A_2 \quad & = 20 \\ X_1 + 2X_2 \quad -S_3 + A_3 & = 19 \\ X_1, X_2, S_1, S_3, A_1, A_2, A_3 & \geq 0. \end{aligned}$$

Basic	W	X_1	X_2	S_1	A_1	A_2	S_3	A_3	RHS
W	1	0	0	0	-1	-1	0	-1	0
A_1	0	1	0	-1	1	0	0	0	4
A_2	0	2	1	0	0	1	0	0	20
A_3	0	1	2	0	0	0	-1	1	19

Eq.0 = Eq.0 +(Eq.1+Eq.2+Eq.3) yields the canonical form below so that we can now start applying the simplex method

Basic	W	X_1	X_2	S_1	A_1	A_2	S_3	A_3	RHS
W	1	4	3	-1	0	0	-1	0	43
A_1	0	<u>**1**</u>	0	-1	1	0	0	0	4
A_2	0	2	1	0	0	1	0	0	20
A_3	0	1	2	0	0	0	-1	1	19

Basic	W	X_1	X_2	S_1	A_1	A_2	S_3	A_3	RHS
W	1	0	3	3	-4	0	-1	0	27
X_1	0	1	0	-1	1	0	0	0	4
A_2	0	0	1	2	-2	1	0	0	12
A_3	0	0	<u>**2**</u>	1	-1	0	-1	1	15

Basic	W	X_1	X_2	S_1	A_1	A_2	S_3	A_3	RHS
W	1	0	0	1.5	-2.5	0	0.5	-1.5	4.5
X_1	0	1	0	-1	1	0	0	0	4
A_2	0	0	0	<u>**1.5**</u>	-1.5	1	0.5	-0.5	4.5
X_2	0	0	1	0.5	-0.5	0	-0.5	0.5	7.5

Basic	W	X_1	X_2	S_1	A_1	A_2	S_3	A_3	RHS
W	1	0	0	0	-1	-1	-10/3	-1	0
X_1	0	1	0	0	0	2/3	1/3	-1/3	7
S_1	0	0	0	1	-1	2/3	1/3	-1/3	3
X_2	0	0	1	0	0	-1/3	-2/3	2/3	6

Optimal phase 1 objective=0, so that means we can move on to Phase 2: Drop the columns for artificial variables, remove the Phase 1 objective, and replace it with the original one.

Phase 2

Basic	Z	X_1	X_2	S_1	S_3	RHS
Z	1	-30	-20	0	0	0
X_1	0	1	0	0	1/3	7
S_1	0	0	0	1	1/3	3
X_2	0	0	1	0	-2/3	6

Put into canonical form: Eq. 0 = Eq. 0 + 30*Eq. 1 + 20* Eq. 3

Basic	Z	X_1	X_2	S_1	S_3	RHS
Z	1	0	0	0	-10/3	330
X_1	0	1	0	0	1/3	7
S_1	0	0	0	1	1/3	3
X_2	0	0	1	0	-2/3	6

No further iterations required - OPTIMAL SOLUTION!

Question 4 *(Q6, p. 213)*

Big-M Method

Putting into standard form and adding artificial variables where needed yields:

$$
\begin{array}{llr}
\text{Maximize } Z = X_1 + X_2 - \text{M}A_1 & & \\
\text{st} & 2X_1 + X_2 - S_1 + A_1 & = 3 \\
 & 3X_1 + X_2 + S_2 & = 3.5 \\
 & X_1 + X_2 + S_3 & = 1 \\
 & X_1, X_2, S_1, S_2, S_3, A_1 \geq 0. &
\end{array}
$$

Basic	Z	X_1	X_2	S_1	A_1	S_2	S_3	RHS
Z	1	-1	-1	0	M	0	0	0
A_1	0	2	1	-1	1	0	0	3
S_2	0	3	1	0	0	1	0	3.5
S_3	0	1	1	0	0	0	1	1

Eq.0 = Eq.0 - M(Eq.1) yields the canonical form below:

Basic	Z	X_1	X_2	S_1	A_1	S_2	S_3	RHS
Z	1	-2M-1	-M-1	M	0	0	0	-3M
A_1	0	2	1	-1	1	0	0	3
S_2	0	3	1	0	0	1	0	3.5
S_3	0	**1**	1	0	0	0	1	1

Basic	Z	X_1	X_2	S_1	A_1	S_2	S_3	RHS
Z	1	0	M	M	0	0	2M+1	-M+1
A_1	0	0	-1	-1	1	0	-2	1
S_2	0	0	-2	0	0	1	-3	0.5
X_1	0	1	1	0	0	0	1	1

This is an optimal tableau (no negative reduced costs in Row 0). However, the artificial variable A_1 is still in the basis at a positive value of 1. Therefore the original problem is **infeasible**.

TWO-PHASE Method:

The constraints in standard form are the same as with the Big-M method. The Phase 1 objective is to

Minimize $W = A_1$

Basic	W	X_1	X_2	S_1	A_1	S_2	S_3	RHS
W	1	0	0	0	-1	0	0	0
A_1	0	2	1	-1	1	0	0	3
S_2	0	3	1	0	0	1	0	3.5
S_3	0	1	1	0	0	0	1	1

Eq.0 = Eq.0 + (Eq.1) yields the canonical form below:

Basic	W	X_1	X_2	S_1	A_1	S_2	S_3	RHS
W	1	2	1	-1	0	0	0	3
A_1	0	2	1	-1	1	0	0	3
S_2	0	3	1	0	0	1	0	3.5
S_3	0	**1**	1	0	0	0	1	1

Basic	Z	X_1	X_2	S_1	A_1	S_2	S_3	RHS
Z	1	0	-1	-1	0	0	-1	1
A_1	0	0	-1	-1	1	0	-2	1
S_2	0	0	-2	0	0	1	-3	0.5
X_1	0	1	1	0	0	0	1	1

This is an optimal tableau for the Phase 1 problem and the Phase 1 objective is >0 (with a positive artificial variable in the basis). This indicates the original problem is **infeasible.**

Question 5 (*Q18, p. 214-215*)

g) First we **require**: $b \geq 0$, c_1 and $c_2 \geq 0$ so that the current solution is optimal. There are several ways that this could lead to alternative optima – any of these would work:
   - If $c_1=0$, we can definitely pivot x_1 into the basis to obtain an alternative optimum (since we can always take the ratio $b/4$ in the first constraint row); a_3 could be any value.
   - If $c_1>0$ and $c_2=0$, then as long as $a_1>0$ we can definitely pivot x_2 into the basis to get an alternative optimum.
   - If c_1, $c_2 > 0$, as long as $a_2>0$ we can also pivot x_5 into the basis to get an alternative optimum.

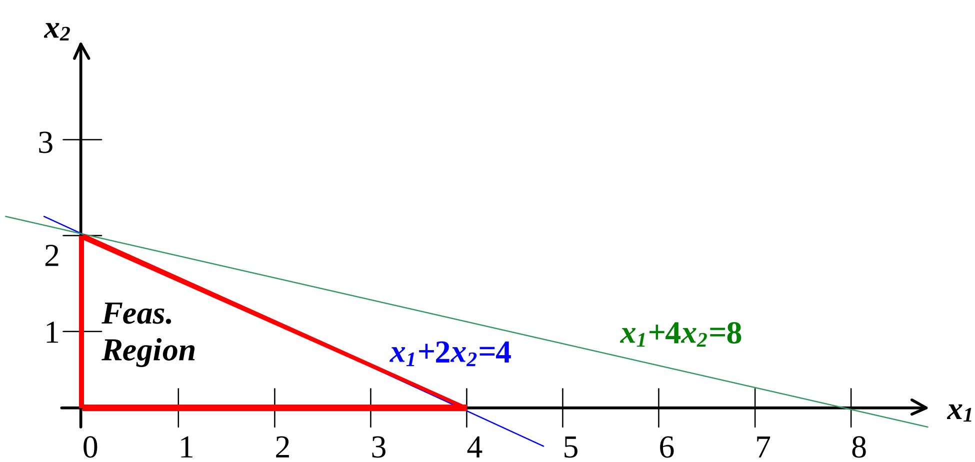
b) This only requires $b<0$, other unknowns could be anything. If this happens there's obviously something wrong with our math, or we chose the wrong row at the previous iteration when doing the minimum ratio test.

c) This only requires $b=0$, other unknowns could be any value.

d) Feasibility requires $b \geq 0$. We need $c_2 < 0$ and $a_1 \leq 0$ for unboundedness through an infinite increase in the value of x_2; c_1, a_2 and a_3 could be anything.

e) Feasibility requires $b \geq 0$. To improve the objective value by bringing x_1 into the basis, we require $c_1<0$ and for x_1 to replace x_6 (the third basic variable in the tableau): we need $a_3 \geq 0$ and we need Row 3 to win the ratio test, i.e., $3/a_3 \leq b/4$

Question 6



Basic	Z	x_1	x_2	S_1	S_2	RHS	
Z	1	-6	-9	0	0	0	
S_1	0	1	**4**	1	0	8	8/4=**2**
S_2	0	1	2	0	1	4	4/2=2

x_2 enters and replaces S_1

Basic	Z	x_1	x_2	S_1	S_2	RHS	
Z	1	-3.75	0	2.25	0	18	
x_2	0	0.25	1	0.25	0	2	2/0.25=8
S_2	0	**0.5**	0	-0.5	1	0	0/0.5=**0**

x_1 enters and replaces S_2

Basic	Z	x_1	x_2	S_1	S_2	RHS	
Z	1	0	0	-1.5	7.5	18	
x_2	0	0	1	**0.5**	-0.5	2	2/0.25=**8**
x_1	0	1	0	-1	2	0	∞

S_1 enters and replaces x_2

Basic	Z	x_1	x_2	S_1	S_2	RHS
Z	1	0	3	0	6	24
S_1	0	0	2	1	-1	4
x_1	0	1	2	0	1	4

OPTIMAL!

Note that corresponding to each of the extreme points (0,0) and (4,0) there is exactly one basic feasible solution. However, corresponding to the extreme point (0,2) we have **three** basic feasible solutions. This is because at the first two points exactly two lines are intersecting while at the third there are **three** lines that intersect: $x_1=0$, $x_1+4x_2=8$ and $x_1+2x_2=4$.

Since $n=4$ and $m=2$, at each BFS we have n-m=2 nonbasic and m=2 basic variables so that at the extreme point (0,2) we have the following three different BFS:

BFS No.	**Nonbasic Variables**	**Basic Variables**	**Intersection of**
1	$x_1=S_1=0$	$x_2=2$, $S_2=0$	x_2 axis and Constr. 1
2	$x_1=S_2=0$	$x_2=2$, $S_1=0$	x_2 axis and Constr. 2
3	$S_1=S_2=0$	$x_2=2$, $x_1=0$	Constr. 2 and Constr. 1

In using the given rule with the simplex method, we first go from (0,0) to (0,2) and improve the objective from 0 to 18. The current BFS corresponds to No. 1 above (x_2 and S_2 basic). At the next iteration we go from BFS no.1 to BFS no. 3 (with x_2 and x_1 basic). However we are still at the same extreme point (0,2) and there is no improvement in the objective at this step. At the next iteration we move to extreme point (4,0) which is the optimum solution with a value of 24.