I.E. 2001 OPERATIONS RESEARCH

(Spring 2012: Solutions to Assignment 4)

(Solutions to Assignment 4)

Question I

First add nonnegative slacks or subtract nonnegative excess variables (from the first two constraints) to make all constraints equalities. Then multiply both sides of equations 2 and 3 by the constant -1 so that the RHS is a nonnegative constant for each. These two steps yield:

$$
\begin{array}{lllll}
\text{Max} & -3X_1 + X_2 - 2X_3 + X_4 & & & \\
\text{st} & -4X_1 + X_2 + X_3 & -X_5 & & = 4 \\
 & -3X_1 + X_2 - 2X_3 & & -X_6 & = 6 \\
 & -X_2 - 4X_3 + X_4 & & & = 1 \\
 & 2X_1 - X_2 + X_3 & & & = 0 \\
 & X_1 \text{ UNR}, X_2, X_3, X_4, X_5, X_6 \geq 0 & & &
\end{array}
$$

Next make sure that all variables are nonnegative by defining $X_1=X_1'-X_1''$ where $X_1'\geq 0$ and $X_1''\geq 0$; and obtain the standard form:

$$
\begin{array}{lllll}
\text{Max} & -3X_1' + 3X_1'' + X_2 - 2X_3 + X_4 & & & \\
\text{st} & -4X_1' + 4X_1'' + X_2 + X_3 & -X_5 & & = 4 \\
 & -3X_1' + 3X_1'' + X_2 - 2X_3 & & -X_6 & = 6 \\
 & -X_2 - 4X_3 + X_4 & & & = 1 \\
 & 2X_1' - 2X_1'' - X_2 + X_3 & & & = 0 \\
 & X_1', X_1'', X_2, X_3, X_4, X_5, X_6 \geq 0 & & &
\end{array}
$$

Question II.

The standard form looks as follows:

1) $X_1 + X_2 + S_1 = 16$

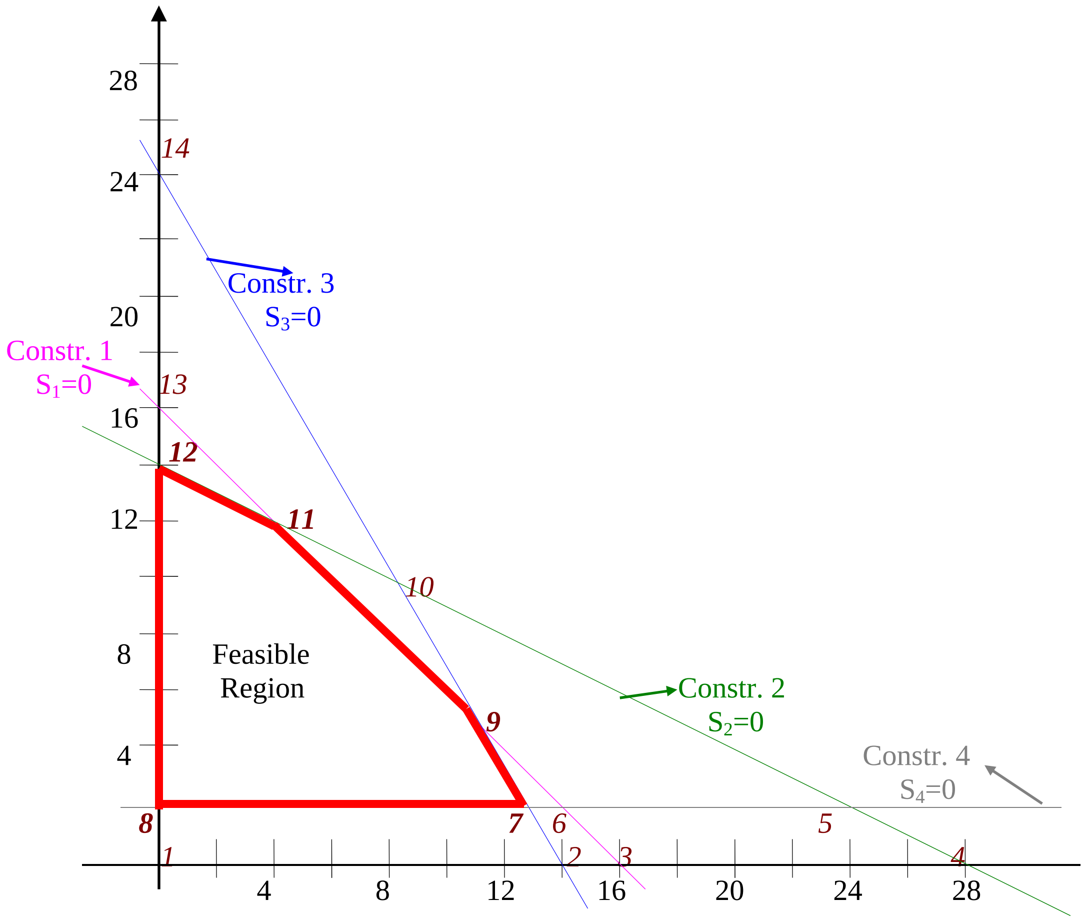
2) $X_1 + 2X_2 + S_2 = 28$

3) $12X_1 + 7X_2 + S_3 = 168$

4) $X_2 - S_4 = 2$ $\qquad X_1, X_2, S_1, S_2, S_3, S_4 \geq 0$

a) The graph is shown below

b) Note that n=6 variables, m=4 constraints. The maximum no. of basic solutions that could possibly exist is $\binom{n}{n-m} = \binom{n}{m} = \binom{6}{4} = 15$. Only 14 of these exist here; there is no basic solution corresponding to setting (X_2, S_4) as nonbasic - notice that constraint 4 is parallel to the X_1-axis (where X_2=0) and the lines corresponding to these do not intersect.



c) The problem has five basic feasible solutions (marked ***7, 8, 9, 11*** and ***12***)

d) The 14 basic solutions are marked on the graph. The variable values at each of these solutions are as follows (Basic **feasible** solutions are in boldface):

Point	Basic Variables	Nonbasic Variables
1	$S_1=16, S_2=28, S_3=168, S_4=-2$	$X_1=X_2=0$
2	$X_1=14, S_1=2, S_2=14, S_4=-2$	$S_3=X_2=0$
3	$X_1=16, S_2=12, S_3=-24, S_4=-2$	$S_1=X_2=0$
4	$X_1=28, S_1=-12, S_3=-168, S_4=-2$	$S_2=X_2=0$
5	$X_1=24, X_2=2, S_1=-10, S_3=-134$	$S_2=S_4=0$
6	$X_1=14, X_2=2, S_2=10, S_3=-14$	$S_1=S_4=0$
7	$\mathbf{X_1=12.833, X_2=2, S_1=1.167, S_2=11.167}$	$\mathbf{S_3=S_4=0}$
8	$\mathbf{X_2=2, S_1=14, S_2=24, S_3=154}$	$\mathbf{X_1=S_4=0}$
9	$\mathbf{X_1=11.2, X_2=4.8, S_2=7.2, S_4=2.8}$	$\mathbf{S_1=S_3=0}$
10	$X_1=8.23, X_2=9.88, S_1=-2.12, S_4=7.88$	$S_2=S_3=0$
11	$\mathbf{X_1=4, X_2=12, S_3=36, S_4=10}$	$\mathbf{S_1=S_2=0}$
12	$\mathbf{X_2=14, S_1=2, S_3=70, S_4=12}$	$\mathbf{S_2=X_1=0}$
13	$X_2=16, S_2=-4, S_3=56, S_4=14$	$S_1=X_1=0$
14	$X_2=24, S_1=-8, S_2=-20, S_4=22$	$S_3=X_1=0$

Question III.

Basic	Z	X_1	X_2	S_1	S_2	S_3	S_4	RHS
Z	1	-2	0	0	2	0	0	56
S_1	0	**0.5**	0	1	-0.5	0	0	2
S_4	0	0.5	0	0	0.5	0	1	12
S_3	0	8.5	0	0	-3.5	1	0	70
X_2	0	0.5	1	0	0.5	0	0	14

The only candidate for entry is X_1 with a negative entry in Equation 0 (also called its *reduced cost*). Conducting the minimum ratio test, Min{2/0.5, 12/0.5, 70/8.5,14/0.5} = Min{**4**,24,8.23,28} = 4, occurs in the first row. Thus S_1 will be replaced in the basis by X_1 at the next iteration. The pivot element is circled and we want a 1 here and zeros elsewhere in the pivot column (i.e., the column for X_1). We perform the following sequence of elementary row operations (*ero*'s):

Row 1 = (Row 1)/0.5
Row 0 = Row 0 + 2*Row 1
Row 2 = Row 2 – 0.5*Row 1
Row 3 = Row 3 – 8.5*Row 1
Row 4 = Row 4 – 0.5*Row 1

This yields:

Basic	Z	X_1	X_2	S_1	S_2	S_3	S_4	RHS
Z	1	0	0	4	0	0	0	64
X_1	0	1	0	2	-1	0	0	4
S_4	0	0	0	-1	1	0	1	10
S_3	0	0	0	-17	5	1	0	36
X_2	0	0	1	-1	1	0	0	12

This is the optimal solution because there are no more negative entries in Row 0. Note that this is **BFS 11** on the graph.

Extra: The unusual thing about this optimal tableau is that while all the basic variables have a reduced cost (entry in Row 0) of zero just like in all the examples seen so far, there is a nonbasic variable (S_2) that also has a **zero** entry in this row. So making S_2 basic by increasing its value will improve the objective by zero units for each 1 unit increase. Thus Z does not get any better, but it also does not get any worse! If we were to do one more iteration and introduce S_2 into the basis (by doing a minimum ratio test on the column for S_2 notice that the leaving variable from the basis would be S_3) we would get an alternative optimum solution (**BFS 9** on the graph). So for this problem the objective function's contour is parallel to the line for Constraint 1 (the pink line)…

Question IV. (Remember that this is a minimization problem!)

ITERATION 1

Basic	Eq.	Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	RHS	
Z	(0)	1	5	4	-1	3	0	0	0	0	
→ S_1	(2)	0	**3**	2	-3	1	1	0	0	24	***24/3***
S_2	(3)	0	3	3	1	3	0	1	0	36	36/3
S_3	(4)	0	-1	2	5	0	0	0	1	3	-

ITERATION 2

Basic	Eq.	Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	RHS	
Z	(0)	1	0	2/3	4	4/3	-5/3	0	0	-40	
X_1	(2)	0	1	2/3	-1	1/3	1/3	0	0	8	-
S_2	(3)	0	0	1	4	2	-1	1	0	12	12/4
→ S_3	(4)	0	0	8/3	**4**	1/3	1/3	0	1	11	***11/4***

ITERATION 3

Basic	Eq.	Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	RHS	
Z	(0)	1	0	-2	0	1	-2	0	-1	-51	
X_1	(2)	0	1	4/3	0	5/12	5/12	0	1/4	43/4	43/4÷5/12
→ S_2	(3)	0	0	-5/3	0	**5/3**	-4/3	1	-1	1	***1÷ (5/3)***
X_3	(4)	0	0	2/3	1	1/12	1/12	0	1/4	11/4	11/4÷1/12

ITERATION 4

Basic	Eq.	Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	RHS
Z	(0)	1	0	-1	0	0	-6/5	-3/5	-2/5	-258/5
X_1	(2)	0	1	7/4	0	0	3/4	-1/4	1/2	21/2
X_4	(3)	0	0	-1	0	**1**	-4/5	3/5	-3/5	3/5
X_3	(4)	0	0	3/4	1	0	3/20	-1/20	3/10	27/10

OPTIMUM TABLEAU.

(Sorry – I didn't realize the numbers would get this messy…)