I.E. 2001 OPERATIONS RESEARCH (Spring 2020)

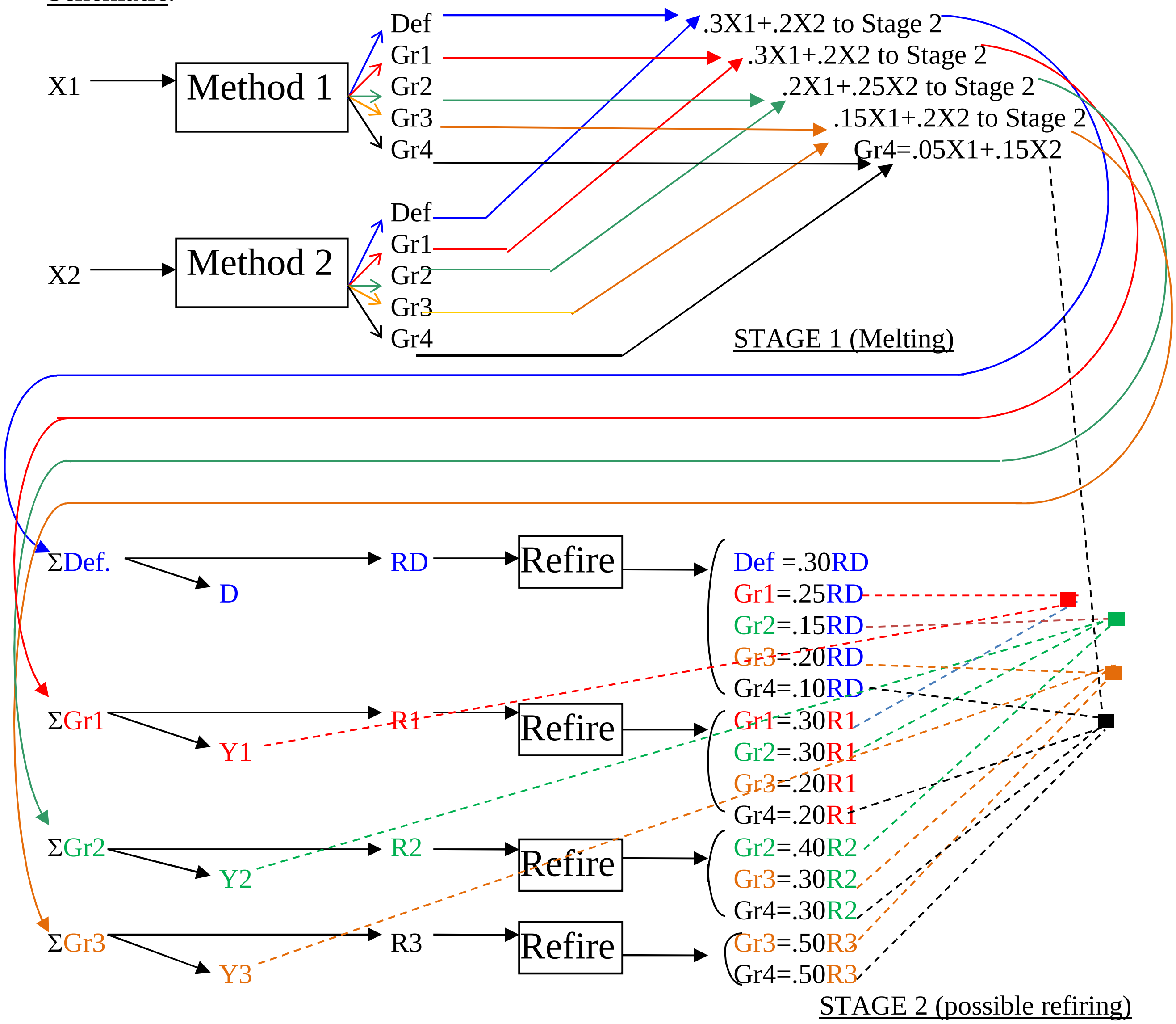
(Solutions to Assignment 3)

Question 1 (Q. 51, p. 122)

Let

X1 = number of transistors' worth of germanium melted by method 1
X2 = number of transistors' worth of germanium melted by method 2
RD = number of defective transistors' worth of germanium refired
R1 = number of grade 1 transistors' worth of germanium refired
R2 = number of grade 2 transistors' worth of germanium refired
R3 = number of grade 3 transistors' worth of germanium refired
D = number of defective transistors' worth of germanium not refired
Y1 = number of grade 1 transistors' worth of germanium not refired
Y2 = number of grade 2 transistors' worth of germanium not refired
Y3 = number of grade 3 transistors' worth of germanium not refired

Schematic:



Then the appropriate LP is

Min z = $(50X1 + 70X2) + (25RD + 25R1 + 25R2 + 25R3)$ Costs
(Melting) (Refiring)

s.t.

$0.3X1 + 0.2X2 = RD + D$
$0.3X1 + 0.2X2 = R1 + Y1$
$0.2X1 + 0.25X2 = R2 + Y2$
$0.15X1 + 0.20X2 = R3 + Y3$ **Material Balance - First Stage (Melting)**

$0.25RD + 0.30R1 + Y1 \geq 3000$ Grade 1 demand
$0.15RD + 0.30R1 + 0.40R2 + Y2 \geq 3000$ Grade 2 demand
$0.20RD + 0.20R1 + 0.30R2 + 0.50R3 + Y3 \geq 2000$ Grade 3 demand
$0.05X1 + 0.15X2 + 0.10RD + 0.20R1 + 0.30R2 + 0.50R3 \geq 1000$ ■ Grade 4 demand
Second Stage (Refire)

$X1 + X2 + RD + R1 + R2 + R3 \leq 20000$ (capacity)

All variables ≥ 0

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<u>Question 52, p. 122</u>

Let

X1 = Tons of purchased boxboard sent through deinking
T1 = Tons of purchased tissue sent through deinking
N1 = Tons of purchased newsprint sent through deinking
B1 = Tons of purchased book paper sent through deinking

X2 = Tons of purchased boxboard sent through asphalt dispersion
T2 = Tons of purchased tissue sent through asphalt dispersion
N2 = Tons of purchased newsprint sent through asphalt dispersion
B2 = Tons of purchased book paper sent through asphalt dispersion

PX = Tons of available boxboard pulp
PT = Tons of available tissue pulp
PN = Tons of available newspaper pulp
PB = Tons of available book paper pulp

PX*i* = Tons of boxboard pulp used for grade *i* paper, *i*=2,3
PT*i* = Tons of tissue pulp used for grade *i* paper, *i*=2,3
PN*i* = Tons of newspaper pulp used for grade *i* paper, *i*=1,3
PB*i* = Tons of book paper pulp used for grade *i* paper, *i*=1,2

A schematic is shown below…
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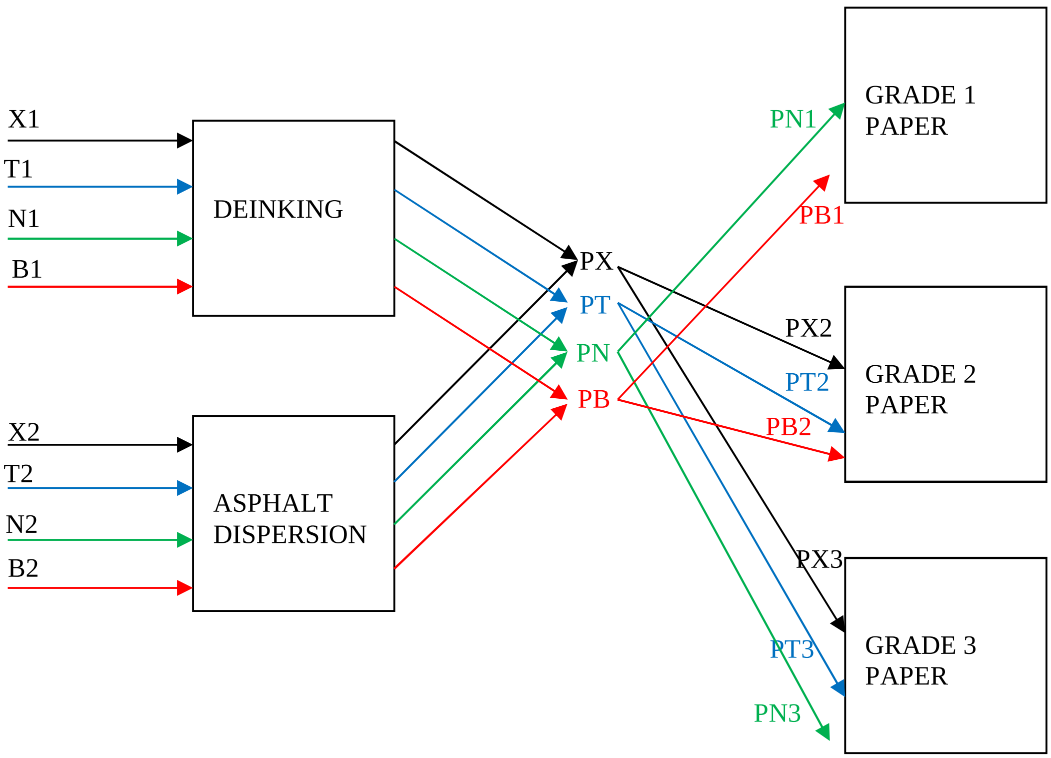


Min Z = $5(X1 + X2) + 6(T1+T2) + 8(N1+N2) + 10(B1+B2)$ (raw materials cost)
$+ 20(X1+T1+N1+B1) + 15(X2+T2+N2+B2)$ (processing cost)

st

$0.9*0.15*X1 + 0.8*0.15*X2 = PX$, i.e. $0.135X1 + 0.12X2 - PX = 0$
$0.9*0.20*T1 + 0.8*0.20*T2 = PT$, i.e. $0.18T1 + 0.16T2 - PT = 0$
$0.9*0.30*N1 + 0.8*0.30*N2 = PN$, i.e. $0.27N1 + 0.24N2 - PN = 0$
$0.9*0.40*B1 + 0.8*0.40*B2 = PB$, i.e. $0.36B1 + 0.32B2 - PB = 0$
(material balance for each pulp type)

$PX2 + PX3 = PX$, i.e. $PX2 + PX3 - PX = 0$
$PT2 + PT3 = PT$, i.e. $PT2 + PT3 - PT = 0$
$PN1 + PN3 = PN$, i.e. $PN2 + PN3 - PN = 0$
$PB1 + PB2 = PB$, i.e. $PB2 + PB3 - PB = 0$
(pulp balance; pulp used = pulp produced)

$PN1 + PB1 \geq 500$
$PX2 + PT2 + PB2 \geq 500$
$PX3 + PT3 + PPN3 \geq 600$ (pulp requirements for paper grades 1, 2, 3)

$X1 + T1 + N1 + B1 \leq 3000$
$X2 + T2 + N2 + B2 \leq 3000$ (processing capacity)

All variables nonnegative

<u>NOTE</u>: You could eliminate the pulp balance constraints altogether and also eliminate the four variables PX, PT, PN, PB by replacing these in the first set of constraints with (PX2+PX3), (PT2+PT3), (PN1+PN3), (PB1+PB2), respectively …

Question 3

a) Define X_j = No. of 100 lb. bags of Type *j* fertilizer used to blend the order for 1000 lbs of *17-14-10* fertilizer.

Then the LP is

$$\begin{array}{lll} \text{Minimize} & 90X_1 + 20X_2 + 30X_3 & \\ \text{St} & 100X_1 + 100X_2 + 100X_3 = 1000 & \text{(total weight must be 1000 lbs.)} \\ & 50X_1 + \quad\quad + 10X_3 \geq 170 & \text{(nitrogen content reqmt.)} \\ & 20X_1 + 15X_2 + 10X_3 \geq 140 & \text{(phosphorus content reqmt.)} \\ & 5X_1 + 20X_2 + 10X_3 \geq 100 & \text{(potassium content reqmt.)} \\ & X_1, X_2, X_3 \geq 0 & \text{(nonnegativity)} \end{array}$$

b) To convert to a two variable problem we can eliminate one of the variables (say X_3) by using the first constraint via X_3=10-X_1-X_2:

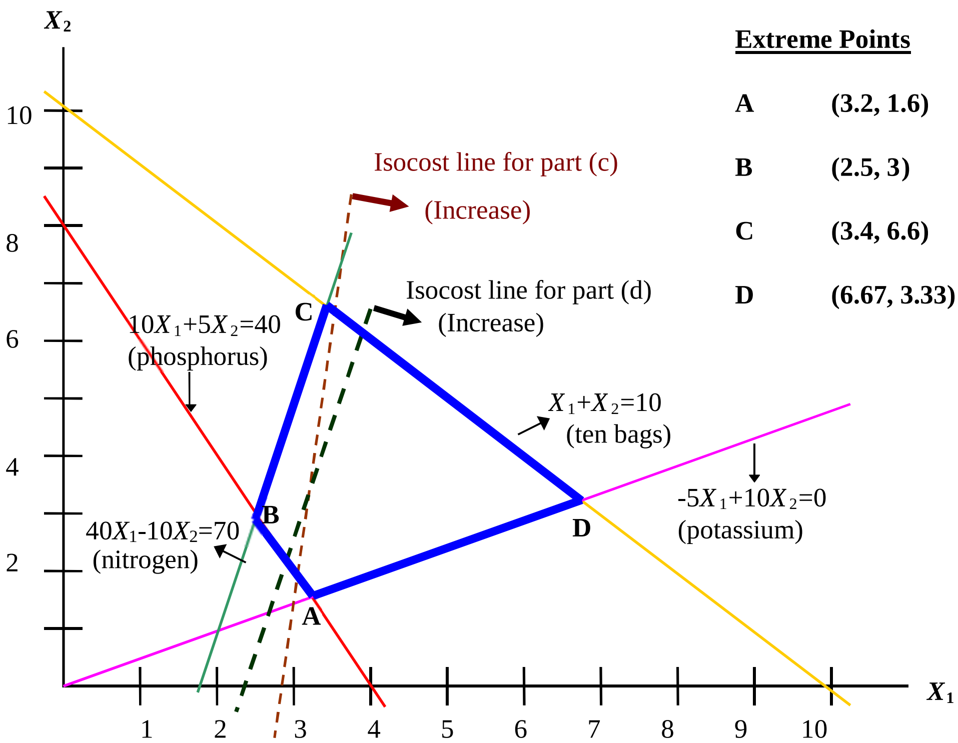
Minimize $90X_1 + 20X_2 + 30(10-X_1-X_2) = 60X_1 - 10X_2 + 300$, i.e.,

Minimize $60X_1 - 10X_2$

st

$$\begin{array}{lcl} 50X_1 + \quad +10(10-X_1-X_2) \geq 170 & \Rightarrow & 40X_1 - 10X_2 \geq 70 \\ 20X_1+15X_2+10(10-X_1-X_2) \geq 140 & \Rightarrow & 10X_1 + 5X_2 \geq 40 \\ 5X_1+20X_2+10(10-X_1-X_2) \geq 100 & \Rightarrow & -5X_1 + 10X_2 \geq 0 \\ X_1, X_2, (10-X_1-X_2) \geq 0 & \Rightarrow & X_1 + X_2 \leq 10, \quad X_1, X_2 \geq 0 \end{array}$$

Note: The value of 300 is a constant that may be factored out of the objective for the modified problem; it just needs to be added on to the optimum value of the modified problem above to get the actual cost for the original.

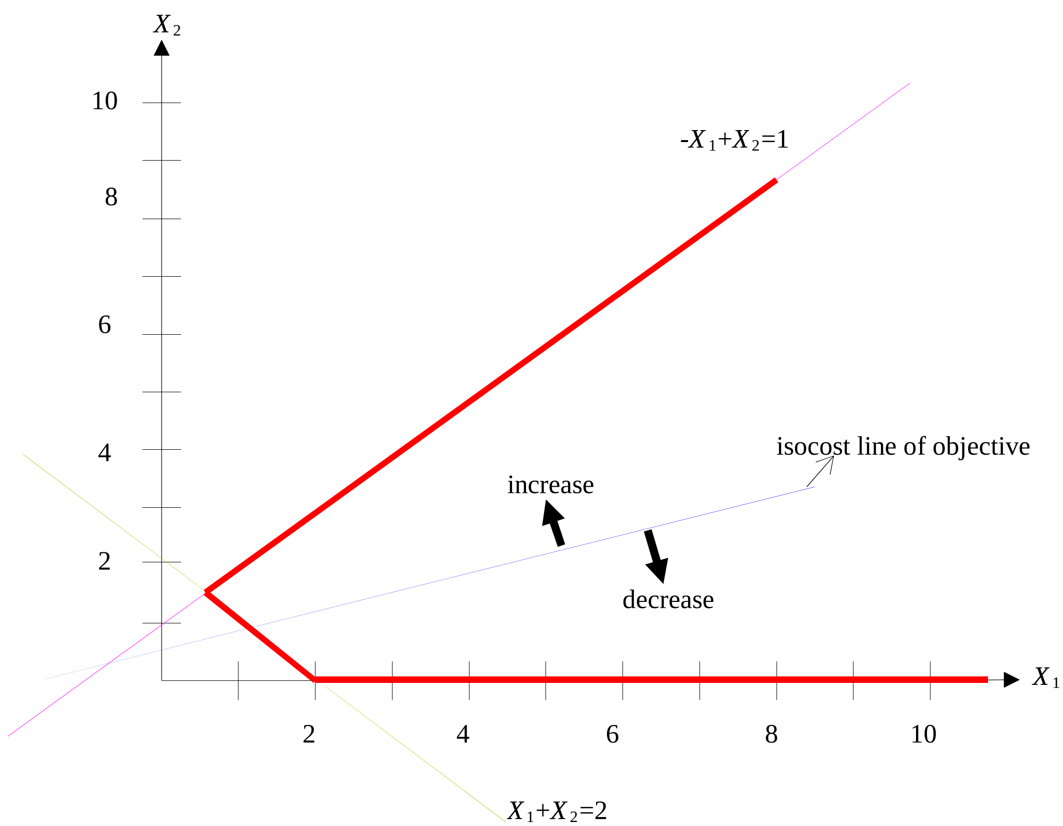


Extreme Points	
A	**(3.2, 1.6)**
B	**(2.5, 3)**
C	**(3.4, 6.6)**
D	**(6.67, 3.33)**

c) From the graph and the slope of the isocost line shown, it is clear that the optimum is at the extreme point B, with X_1=2.5, X_2=3. Note that this implies that X_3 = 10 - (2.5+3) = 4.5 bags. The minimum cost is given by (60*2.5) - (10*3) + 300 = 420, or equivalently, (90*2.5) + (20*3) + (30*4.5) = $420. The constraints on nitrogen and phosphorus are active, and the bag and potassium constraints are inactive (note though, that in terms of the <u>original</u> problem exactly ten bags are used). The minimum nutrient requirements of nitrogen and phosphorus are thus both exactly met, while that of potassium is exceed by [(-5*2.5) + (10*3)] - 0 = 17.5 lbs. (or equivalently, by [(5*2.5) + (20*3) + (10*4.5)] - 100 = 17.5 lbs.).

d) With the new costs the objective is given by 85X_1 + 10X_2 + 25(10-X_1-X_2) = 60X_1 - 15X_2 + 250, i.e., Minimize 60X_1 - 15X_2. From the slope of the isocost line shown, the optimum is along the line joining extreme points B and C. There are two optimum extreme points: (X_1=2.5, X_2=3) and (X_1=3.4, X_2=6.6). The minimum cost in both cases is given by $355 (= 60*2.5-15*3+250 =60*3.4-15*6.6+250). There are infinitely many optimum solutions to the LP and are given in general by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} 2.5 \\ 3 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 3.4 \\ 6.6 \end{bmatrix} = \begin{bmatrix} 3.4-0.9\lambda \\ 6.6-3.6\lambda \end{bmatrix}$, where λ is a constant in the closed interval [0,1]. The objective is given by 60*(3.4-0.9λ) - 15*(6.6-3.6λ) + 250 = 355. In terms of the original problem, we may specify the complete solution set: X_1, X_2 as above and X_3=10-(X_1+X_2) = 10-[(3.4-0.9λ)+(6.6-3.6λ)] = 4.5λ, so that the objective is 85*(3.4-0.9λ) + 10*(6.6-3.6λ) +25*4.5λ = $355. The constraint on nitrogen is always active and if one chooses B then the phosphorus constraint is also active, while if one chooses C then the bag constraint is also active. The potassium constraint is inactive. The minimum nutrient requirement of nitrogen is exactly met. At B the phosphorus requirement is also exactly met while that of potassium is exceed by [(-5*2.5) + (10*3)] - 0 = 17.5 lbs. At C the phosphorus requirement is exceeded by [(10*3.4)+(5*6.6)] - 40 = 27, while that of potassium is exceed by [(-5*3.4) + (10*6.6)] - 0 = 49 lbs. At points between B and C the requirement of phosphorus is exceeded by [10*(3.4-0.9λ) + 5*(6.6-3.6λ)] - 40 = (27-27λ), while that of potassium is exceed by [-5*(3.4-0.9λ) + 10*(6.6-3.6λ)] - 0 = (49-31.5λ) lbs.

Question 3



The feasible region is open and unbounded.

The problem has no optimal solution for minimizing $-2X_1 + 6X_2$, since the slope of the isocost line shown is such that it can be moved in the direction of decrease without ever leaving the feasible region completely: X_1 can be made arbitrarily large while keeping X_2 fixed at some small nonnegative value so that the objective goes to $-\infty$.

If the problem is a maximization of the objective, the problem is still unbounded because the isocost line can also be moved in the direction of increase without ever leaving the feasible region completely: X_2 can be made arbitrarily large while ensuring that $6X_2$ is always larger than $2X_1$ so that the objective goes to ∞.