Dynamics of a Quantum Two-Level System

Consider a quantum system represented by a two basis state wavefunction:

$$\psi(x,t) = c_1(t)\phi_1(x) + c_2(t)\phi_2(x) \qquad [1]$$

Here $\phi_{1,2}(x)$ are time-*in*dependent basis functions , presumed to be orthonormal (and, for simplicity, real valued).

Denote the Hamiltonian governing the motion of the system as $\hat{H}$

Denote the relevant Hamiltonian matrix element as:

$$\int dx\phi_1(x)\hat{H}\phi_1(x) = E_1 \;\; ; \int dx\phi_2(x)\hat{H}\phi_2(x) = E_2$$

$$\int dx\phi_1(x)\hat{H}\phi_2(x) = \Delta = \int dx\phi_2(x)\hat{H}\phi_1(x)$$

… with Δ presumed real-valued.

Now, appeal to the time-dependent Schrodinger Eq.: $i\hbar\frac{\partial\psi(x,t)}{\partial t} = \hat{H}\psi(x,t)$

Substituting in the two-basis expansion representation of the wavefunction on the r.h.s. of Eq [1], and using the orthonormality properties of the basis functions, we obtain the matrix Schrodinger Eq.:

$$i\hbar \begin{pmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \end{pmatrix} = \boldsymbol{H} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} ; \boldsymbol{H} \equiv \begin{bmatrix} \varepsilon & \Delta \\ \Delta & -\varepsilon \end{bmatrix} \qquad [2]$$

Note: Here $\varepsilon = (E_1 - E_2)/2$

… and we have shifted the energy axis by the amount $(E_1 + E_2)/2$ so that the origin of energy at the midpoint of the E_1, E_2 energy gap. [This will not change any physical properties of the isolated two-level system (TLS).]

Symmetric TLS.

Let us first consider the case of a symmetric TLS, i.e., $\varepsilon = 0$

From Eq. [2] we deduce for the symmetric TLS tht:

$$\ddot{c}_1(t) = -(\Delta / \hbar)^2 c_1(t)$$

If we assume the initial values $c_1(0) = 1 \; ; \; c_2(0) = 0$

[At t=0, the system is prepared in basis state 1.]

then: $c_1(t) = \cos(\Delta t / \hbar)$

… or:

$$P_1(t) = |c_1(t)|^2 = \cos^2(\Delta t / \hbar)$$

Symmetric TLS

Asymmetric TLS.

Now consider the case of a symmetric TLS, i.e., $\varepsilon \neq 0$

Using a similar method of analysis, one obtains:

$$P_1(t) = |c_1(t)|^2 = 1 - \frac{\Delta^2}{\Delta^2 + \varepsilon^2} \sin^2(\sqrt{\varepsilon^2 + \Delta^2}\, t / \hbar)$$

Asymmetric TLS

Illustration of dynamics of Symmetric and Asymmetric Two-level Systems:

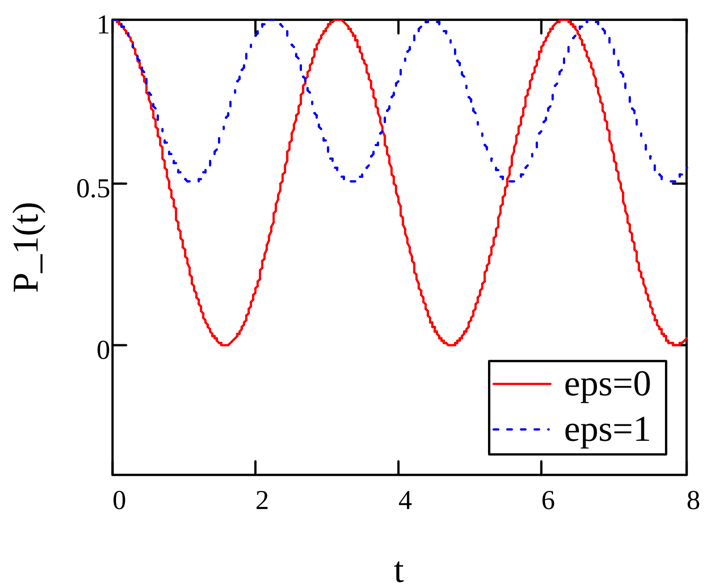

Fig. 2. Probability $P_1(t)$ to be in state *1* as a function of time for a two level system (=TLS), given initial preparation in state *1*, and $\Delta = 1$: solid line, $\varepsilon = 0$ (symmetric TLS); dashed line, $\varepsilon = 1$ (asymmetric TLS). [Note: $\hbar = 1$ here.]