

Chapter 4 – Angular Momentum

$$\vec{l} = \vec{r} \times \vec{p}$$

$$l_x = yp_z - zp_y$$

$$l_y = zp_x - xp_z$$

$$l_z = xp_y - yp_x$$

$$l^2 = l_x^2 + l_y^2 + l_z^2$$

$$l_z = \frac{\hbar}{i} \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

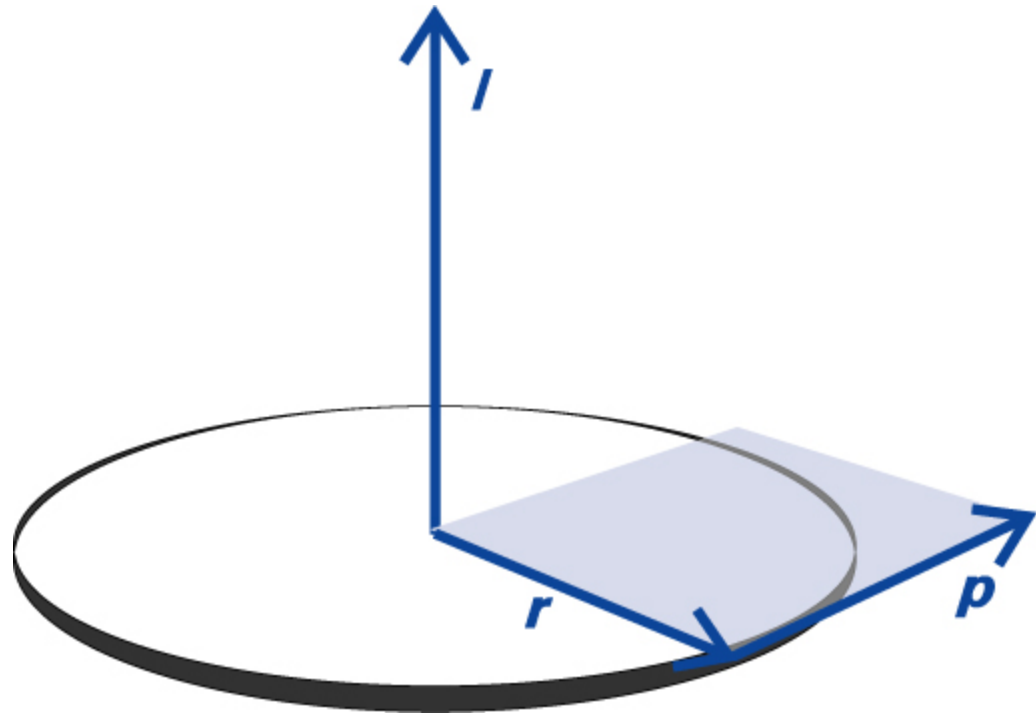
$$[l_x, l_y] = i\hbar l_z$$

$$[l_y, l_z] = i\hbar l_x$$

$$[l_z, l_x] = i\hbar l_y$$

$$[l^2, l_z] = 0$$

Can know l^2 and one component of l , usually taken to be l_z



Shift operators

$$l_+ = l_x + il_y$$

$$l_- = l_x - il_y$$

$$[l^2, l_{\pm}] = 0$$

$$l_z |\lambda, m_\ell\rangle = m_\ell \hbar |\lambda, m_\ell\rangle$$

$$l^2 |\lambda, m_\ell\rangle = \underbrace{f(\ell, m_\ell)}_{\substack{\text{will be } \ell(\ell+1)}} \hbar^2 |\lambda, m_\ell\rangle$$

λ , and m_ℓ are quantum numbers. We will later show that m_ℓ is the quantum number we found previously

$$l^2 - l_z^2 = l_x^2 + l_y^2$$

eigenvalues of this operator are non neg.

$$(l^2 - l_z^2) |\lambda, m_\ell\rangle = [f(\ell, m_\ell) - m_\ell^2] \hbar^2 |\lambda, m_\ell\rangle$$

$$\Rightarrow f \geq m_\ell^2$$

$$\begin{aligned} \ell^2 \ell_+ |\lambda, m_\ell\rangle &= \ell_+ \ell^2 |\lambda, m_\ell\rangle = \ell_+ f(\lambda, m_\ell) \hbar^2 |\lambda, m_\ell\rangle \\ &= f(\lambda, m_\ell) \hbar^2 \ell_+ |\lambda, m_\ell\rangle \end{aligned}$$

ℓ_+ leaves
magnitude of any
momentum
unchanged.

$$\ell_+ |\lambda, m_\ell\rangle = c_+ (f, m_\ell) \hbar |\lambda, m_\ell + 1\rangle \quad \text{step up}$$

$$\ell_- |\lambda, m_\ell\rangle = c_- (f, m_\ell) \hbar |\lambda, m_\ell - 1\rangle \quad \text{step down}$$

Shift operators shift m_ℓ by ± 1 each time they operate

But m_ℓ has a max value, which we call ℓ .

So

$$\ell_+ |\lambda, \ell\rangle = 0 \quad \text{since we can't step past } \ell.$$

$$\ell_- \ell_+ |\lambda, \ell\rangle = 0$$

$$(\ell^2 - \ell_z^2 - \hbar \ell_z) |\lambda, \ell\rangle = 0$$

$$\ell^2 |\lambda, \ell\rangle = (\ell^2 + \ell) \hbar^2 |\lambda, \ell\rangle$$

So we have proven

$$l^2 |l, m_\ell\rangle = l(l+1)\hbar^2 |l, m_\ell\rangle$$

where

$$m_\ell = -l, -l+1, \dots, l-1, l$$

From the algebra of the operators, i.e.,
without reference to the S. E.

$$l = 0 \quad m_\ell = 0$$

$$l = \frac{1}{2} \quad m_\ell = -\frac{1}{2}, \frac{1}{2}$$

$$l = 1 \quad m_\ell = -1, 0, 1$$

$$l = \frac{3}{2} \quad m_\ell = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$$

$$l = 2 \quad m_\ell = -2, -1, 0, 1, 2$$

Cannot have other values of l
would not give a symmetrical ladder

We know from the rigid rotor problem that $\ell = 0, 1, 2, \dots$

The values $\ell = \frac{1}{2}, \frac{3}{2}$ etc. actually come about due to spin

Quantum numbers s, m_s

In general j, m_j

$$j_{\pm} |j, m_j\rangle = c_{\pm}(j, m_j) |j, m_j \pm 1\rangle$$

$$c_{+}(j, m_j) = \sqrt{j(j+1) - m_j(m_j + 1)}$$

$$c_{-}(j, m_j) = \sqrt{j(j+1) - m_j(m_j - 1)}$$

Can use operators to prove

$$\psi_{\ell,\ell} = N \sin^{\ell} e^{i\ell\phi}$$

can use step down operator to generate other functions

Spin – Uhlenbeck & Goudsmit (1925)

$$s = \frac{1}{2}, \quad m_s = +\frac{1}{2}, -\frac{1}{2}$$

Dirac showed how spin derives
from relativistic QM

Explained the Stern-Gerlach experiment

$$\alpha = \left| \frac{1}{2}, \frac{1}{2} \right\rangle \quad \beta = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$s_z \alpha = \frac{\hbar}{2} \alpha \quad s_+ \alpha = 0$$

$$s_z \beta = -\frac{\hbar}{2} \beta \quad s_- \beta = 0$$

$$s^2 \alpha = \frac{3}{4} \hbar^2 \alpha \quad s_+ \beta = \hbar \alpha$$

$$s^2 \beta = \frac{3}{4} \hbar^2 \beta \quad s_- \alpha = \hbar \beta$$

Matrix repr.

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left. \begin{aligned} \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \sigma_+ &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, & \sigma_- &= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \\ S_q &= \frac{1}{2} \hbar \sigma_q, & q &= x, y, z, +, - \end{aligned} \right] \text{Pauli matrices}$$

Coupling angular momenta

$$\mathbf{j}_1, \mathbf{j}_2 \xrightarrow{\text{combine}} \mathbf{j}$$

$$|j_1, m_1\rangle, |j_2, m_2\rangle \longrightarrow |j_1, m_1; j_2, m_2\rangle \quad \text{One basis}$$

$$[j_{1q}, j_{2q'}] = 0$$

$j_1^2, j_{1z}, j_2^2, j_{2z}$ all commute with one another

$$\text{Now, consider } \vec{j} = \vec{j}_1 + \vec{j}_2$$

$$[j_x, j_y] = i\hbar j_z, \text{ etc.}$$

so \vec{j} is an angular momentum.

J must have magnitude $\sqrt{j(j+1)}\hbar$

With j being integral or half integral

j_z has value $m_j\hbar$, $m_j = -j, j-1, \dots, j$

$$[j^2, j_1^2] = 0, \quad [j^2, j_2^2] = 0$$

$$[j_{1z}, j^2] = 2i\hbar [j_{1y}j_{2x} - j_{1x}j_{2y}]$$

cannot specify m_{j_1} or m_{j_2} if we specify j

uncoupled $|j_1, m_{j_1}; j_2, m_{j_2}\rangle$

or
coupled $|j_1, j_2; j, m_j\rangle$

$$j_z = j_{1z} + j_{2z} \quad \text{so} \quad m_j = m_{j_1} + m_{j_2}$$

$$\text{total \# states} \quad (2j_1 + 1)(2j_2 + 1)$$

$$j = j_1 + j_2, j_1 + j_2 - 1, \dots, |j_1 - j_2|$$

$$\text{if } j_1 = \frac{1}{2} \text{ and } j_2 = \frac{3}{2} \Rightarrow j = 2, 1$$

$$s_1 = \frac{1}{2}, s_2 = \frac{1}{2} \quad \alpha_1\alpha_2, \alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2$$

$$\Rightarrow S = 1, 0$$

$$\left\{ \begin{array}{l} |1, 1\rangle = \alpha_1\alpha_2 \\ |1, 0\rangle = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 + \beta_1\alpha_2) \\ |1, -1\rangle = \beta_1\beta_2 \\ |0, 0\rangle = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2) \end{array} \right.$$

One singlet state and three components of the spin triplet state

two p electrons

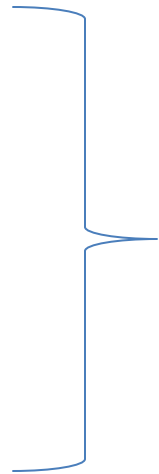
S, P, D states

three p electrons

$S \rightarrow P$

$P \rightarrow S, P, D$

$D \rightarrow F, P, D$



$F, 2D, 3P, S$

27 arrangements,
not counting spin

couple two
electrons

couple in
third
electron