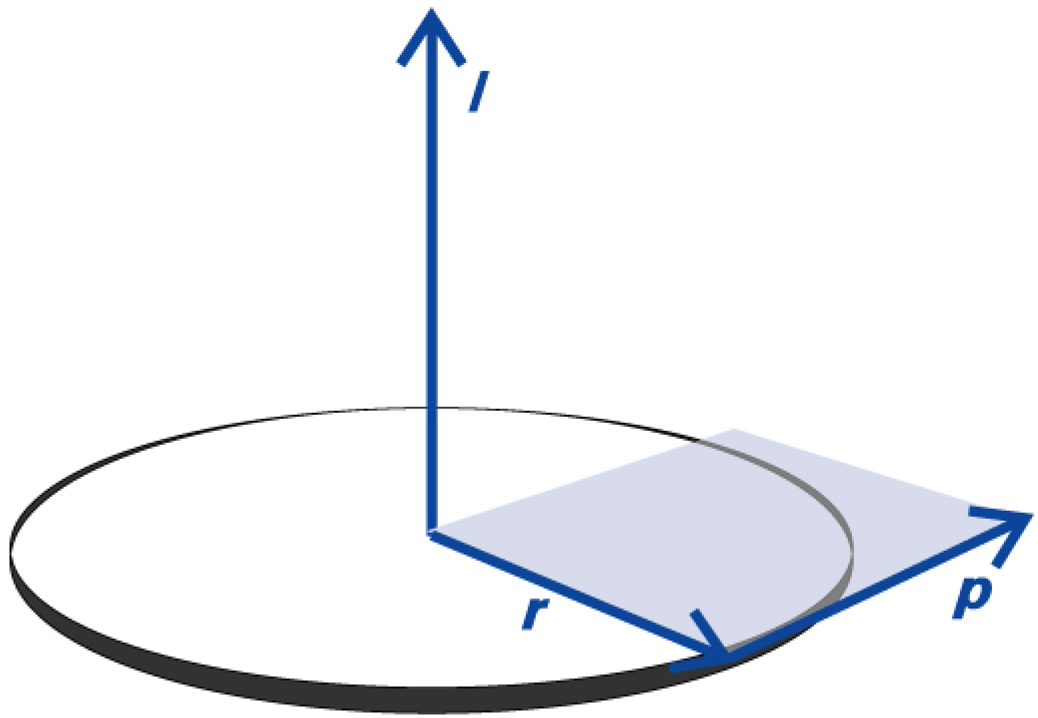
Chapter 4 – Angular Momentum

$$\vec{\ell} = \vec{r} x \vec{p}$$



$$\ell_x = yp_z - zp_y$$

$$\ell_y = zp_x - xp_z$$

$$\ell_z = xp_y - yp_x$$

$$\ell^2 = \ell_x^2 + \ell_y^2 + \ell_z^2$$

$$\ell_z = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

$$\left[\ell_x, \ell_y \right] = i\hbar \ell_z$$

$$\left[\ell_y, \ell_z \right] = i\hbar \ell_x$$

$$\left[\ell_z, \ell_x \right] = i\hbar \ell_y$$

$$\left[\ell^2, \ell_z \right] = 0$$

Can know ℓ^2 and one component of ℓ , usually taken to be ℓ_z

Shift operators

$$\ell_{+} = \ell_{x} + i\ell_{y}$$

$$\ell_{-} = \ell_{x} - i\ell_{y}$$

$$\left[\ell^{2}, \ell_{\pm}\right] = 0$$

$$\ell_{z}\left|\lambda, m_{\ell}\right\rangle = m_{\ell}\hbar\left|\lambda, m_{\ell}\right\rangle$$

$$\ell^{2}\left|\lambda, m_{\ell}\right> = f\left(\ell, m_{\ell}\right)\hbar^{2}\left|\lambda, m_{\ell}\right>$$

$\lambda, and\ m_{\ell}$ are quantum numbers. We will later show that m_{ℓ} is the quantum number we found previously

$f(\ell, m_\ell)$ will be $\ell\left(\ell+1\right)$

$$\ell^{2} - \ell_{z}^{2} = \ell_{x}^{2} + \ell_{y}^{2}$$

eigenvalues of this operator are non neg.

$$\left(\ell^{2} - \ell_{z}^{2}\right)\left|\lambda, m_{\ell}\right\rangle = \left[f\left(\ell, m_{\ell}\right) - m_{\ell}^{2}\right]\hbar^{2}\left|\lambda, m_{\ell}\right\rangle$$

$$\Rightarrow f \geq m_{\ell}^{2}$$

$$\ell^2 \ell_+ \left|\lambda, m_e\right\rangle = \ell_+ \ell^2 \left|\lambda, m_\ell\right\rangle = \ell_+ f\left(\lambda, m_\ell\right)\hbar^2 \left|\lambda, m_\ell\right\rangle$$
$$= f\left(\lambda, m_\ell\right)\hbar^2 \ell_+ \left|\lambda, m_\ell\right\rangle$$

ℓ_+ leaves magnitude of any momentum unchanged.

$$\ell_+ \left|\lambda, m_\ell\right\rangle = c_+\left(f, m_\ell\right)\hbar\left|\lambda, m_\ell + 1\right\rangle$$ step up

$$\ell_- \left|\lambda, m_\ell\right\rangle = c_-\left(f, m_\ell\right)\hbar\left|\lambda, m_\ell - 1\right\rangle$$ step down

Shift operators shift m_ℓ by ± 1 each time they operate

But m_ℓ has a max value, which we call ℓ .

So

$$\ell_+ \left|\lambda, \ell\right\rangle = 0$$ since we can't step past ℓ .

$$\ell_- \ell_+ \left|\lambda, \ell\right\rangle = 0$$
$$\left(\ell^2 - \ell_z^2 - \hbar \ell_z\right)\left|\lambda, \ell\right\rangle = 0$$
$$\ell^2 \left|\lambda, \ell\right\rangle = \left(\ell^2 + \ell\right)\hbar^2 \left|\lambda, \ell\right\rangle$$

So we have proven

$$\ell^2 \left| \ell, m_\ell \right\rangle = \ell\left(\ell+1\right)\hbar^2 \left| \ell, m_\ell \right\rangle$$

where

$$m_\ell = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

From the algebra of the operators, i.e., without reference to the S. E.

$\ell = 0 \qquad m_\ell = 0$

$\ell = \frac{1}{2} \qquad m_\ell = -\frac{1}{2}, \frac{1}{2}$

$\ell = 1 \qquad m_\ell = -1, 0, 1$

$\ell = \frac{3}{2} \qquad m_\ell = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$\ell = 2 \qquad m_\ell = -2, -1, 0, 1, 2$

Cannot have other values of ℓ would not give a symmetrical ladder

We know from the rigid rotor problem that $\ell = 0, 1, 2, \ldots$

The values $\ell = \frac{1}{2}, \frac{3}{2}$ *etc.* actually come about due to spin

Quantum numbers *s*, m_s

In general *j*, m_j

$$j_{\pm} \left| j, m_j \right\rangle = c_{\pm}\left(j, m_j \right) \left| j, m_j \pm 1 \right\rangle$$

$$c_{+}\left(j, m_j \right) = \sqrt{j\left(j+1 \right) - m_j \left(m_j + 1 \right)}$$

$$c_{-}\left(j, m_j \right) = \sqrt{j\left(j+1 \right) - m_j \left(m_j - 1 \right)}$$

Can use operators to prove

$$\psi_{\ell,\ell} = N \sin^{\ell} e^{i\ell\phi}$$

can use step down operator to generate other functions

Spin – Uhlenbeck & Goudsmit (1925)

$$s = \frac{1}{2}, \quad m_s = +\frac{1}{2}, -\frac{1}{2}$$

Dirac showed how spin derives from relativistic QM

Explained the Stern-Gerlach experiment

$$\alpha = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \qquad \beta = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$s_z\alpha = \frac{\hbar}{2}\alpha \qquad s_+\alpha = 0$$

$$s_z\beta = -\frac{\hbar}{2}\beta \qquad s_-\beta = 0$$

$$s^2\alpha = \frac{3}{4}\hbar^2\alpha \qquad s_+\beta = \hbar\alpha$$

$$s^2\beta = \frac{3}{4}\hbar^2\beta \qquad s_-\alpha = \hbar\beta$$

Matrix repr.

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

Pauli matrices

$$S_q = \frac{1}{2}\hbar\sigma_q, \quad q = x, y, z, +, -$$

Coupling angular momenta

$$j_1, \quad j_2 \xrightarrow{\text{combine}} j$$

$$|j_1, m_1\rangle, \quad |j_2, m_2\rangle \longrightarrow |j_1, m_1; j_2, m_2\rangle$$ One basis

$$\left[j_{1q}, j_{2q'} \right] = 0$$

$j_1^2, j_{1z}, j_2^2, j_{2z}$ all commute with one another

Now, consider $\vec{j} = \vec{j}_1 + \vec{j}_2$

$$\left[j_x, j_y \right] = i\hbar j_z,$$ etc.

so $\vec{j}$ is an angular momentum.

J must have magnitude $\sqrt{j(j+1)}\hbar$

With *j* being integral or half integral

j_z has value $m_j\hbar, \quad m_j = -j, j-1, \dots j$

$$\left[j^2, j_1^2\right] = 0, \qquad \left[j^2, j_2^2\right] = 0$$

$$\left[j_{1z}, j^2\right] = 2i\hbar\left[j_{1y}j_{2x} - j_{1x}j_{2y}\right]$$

cannot specify m_{j1} or m_{j2} if we specify *j*

uncoupled $\left|j_1, m_{j_1}; j_2, m_{j_2}\right\rangle$

or
coupled $\left|j_1, j_2; j, m_j\right\rangle$

$$j_z = j_{1z} + j_{2z} \quad \text{so} \quad m_j = m_{j_1} + m_{j_2}$$

total # states $\left(2j_1+1\right)\left(2j_2+2\right)$

$$j = j_1 + j_2, \quad j_1 + j_2 - 1, \ldots, \left| j_1 - j_2 \right|$$

if $j_1 = \frac{1}{2}$ and $j_2 = \frac{3}{2} \Rightarrow j = 2,1$

$$s_1 = \frac{1}{2}, s_2 = \frac{1}{2} \qquad \alpha_1\alpha_2, \alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2$$

$$\Rightarrow S = 1,0$$

$$\left.\begin{array}{l} \left|1,1\right\rangle = \alpha_1\alpha_2 \\ \left|1,0\right\rangle = \frac{1}{\sqrt{2}}\left(\alpha_1\beta_2 + \beta_1\alpha_2\right) \\ \left|1-1\right\rangle = \beta_1\beta_2 \end{array}\right\}$$

$$\left|0,0\right\rangle = \frac{1}{\sqrt{2}}\left(\alpha_1\beta_2 - \beta_1\alpha_2\right)$$

One singlet state and three components of the spin triplet state two *p* electrons

S, P, D states

three *p* electrons

$$S \rightarrow P$$

$$P \rightarrow S, P, D$$

$$D \rightarrow F, P, D$$

couple two electrons

couple in third electron

$$F, 2D, 3P, S$$

27 arrangements, not counting spin