$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{-m^2}{\sin^2\theta}\right)$$

$L^2 S = cS$ suppose $S = \sin\theta$

$$\frac{\partial^2}{\partial\theta^2}\sin\theta = -\sin\theta$$

$$\cot\theta\frac{\partial}{\partial\theta}\sin\theta = \frac{\cos^2\theta}{\sin\theta}$$

$$\frac{-m^2}{\sin^2\theta}\sin\theta = \frac{-m^2}{\sin\theta}$$

$$-\hbar^2\left[-\sin\theta + \frac{\cos^2}{\sin\theta} - \frac{m^2}{\sin\theta}\right] = c\sin\theta$$

$$-\hbar^2\left[-\sin\theta + \frac{\cos^2\theta - m^2}{\sin\theta}\right] = c\sin\theta$$

if $m = \pm 1 \rightarrow -\hbar^2[-\sin\theta - \sin\theta] = c\sin\theta$

$c = 2\hbar^2$

suppose $S = \cos\theta$

If you substitute this into the equation you will find that $m$ has to be zero, and that the eigenvalue is the same.

$P_0 = 1$ $\quad 1$
$P_1 = x$ $\quad \cos\theta$
$P_2 = \frac{1}{2}(3x^2-1)$ $\quad (3\cos^2\theta - 1)$
$P_3 = \frac{1}{2}(5x^3-3x)$ $\quad (5\cos^3\theta - 3\cos\theta)$

Legendre polynomials

---

Associated Legendre Polynomials:

$P_0^0 = 1$

$P_1^0 = x$, $P_1^1 = -\sqrt{1-x^2}$

$P_2^0 = (3x^2-1)$, $P_2^1 = -3x\sqrt{1-x^2}$, $P_2^2 = 1-x^2$

Are $\cos\theta$ and $(3\cos^2\theta - 1)$ orthogonal

$$\int_0^{\pi} (\cos\theta)(3\cos^2\theta - 1)\sin\theta\, d\theta$$

$$= \int_0^{\pi} (3\cos^3\theta - \cos\theta)(-d(\cos\theta))$$

$$= \frac{3\cos^4\theta}{4} - \frac{\cos^2\theta}{2} \Big|_0^{\pi} = 0$$