Operator approach to the Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0{}^2\hat{x}^2$$

Change of variables

$\hat{\zeta} = \beta\hat{x}$ where $\beta = \sqrt{\frac{m\omega_0}{\hbar}}$ dimensionless position

$$\hat{\pi} = \frac{\beta}{m\omega_0}\hat{p}$$

dimensionless momentum

$$\hat{H} = \frac{\hbar\omega_0}{2}\left[\hat{\pi}^2 + \hat{\zeta}^2\right]$$

$$\hat{h} = \frac{1}{\hbar\omega_0}\hat{H} = \frac{1}{2}\left[\hat{\pi}^2 + \hat{\zeta}^2\right]$$

$$\hat{H}\,|\,\psi\rangle = E\,|\,\psi\rangle \;\; \Rightarrow \hat{h}\,|\,\psi\rangle = \varepsilon\,|\,\psi\rangle$$

$\varepsilon = \frac{E}{\hbar\omega_0}$ dimensionless energy

$$[\hat{x}, \hat{p}] = i\hbar, \quad [\hat{\zeta}, \hat{\pi}] = i$$

$$[\hat{\zeta} + i\hat{\pi}][\hat{\zeta} - i\hat{\pi}] = \hat{\zeta}^2 + \hat{\pi}^2 - 1$$

So $\hat{h} = \left(\frac{\hat{\zeta} - i\hat{\pi}}{\sqrt{2}}\right)\left(\frac{\hat{\zeta} + i\hat{\pi}}{\sqrt{2}}\right) + \frac{1}{2}$

raising operator $\hat{a}^{\dagger} = \left(\frac{\hat{\zeta} - i\hat{\pi}}{\sqrt{2}}\right)$

lowering operator $\hat{a} = \left(\frac{\hat{\zeta} + i\hat{\pi}}{\sqrt{2}}\right)$

$\hat{a}$ and $\hat{a}^{\dagger}$ are Hermitian conjugates of one another

$$[\hat{a}, \hat{a}^{\dagger}] = 1$$

$$\hat{a}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p})$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^{\dagger} + \hat{a})$$

$$\hat{p} = \sqrt{\frac{\hbar m\omega}{2}}i(\hat{a}^{\dagger} - \hat{a})$$

So $\hat{h} = \hat{a}^\dagger \hat{a} + \frac{1}{2}$

Can show that $\hat{a}^\dagger \,|\, n\rangle = \sqrt{n+1}\,|\, n+1\rangle$

where $|\, n\rangle$ and $|\, n+1\rangle$ are the n^{th} and the $(n+1)^{th}$ eigenfunctions of $\hat{h}$

Also $\hat{a}\,|\, n\rangle = \sqrt{n}\,|\, n-1\rangle$

Note $\hat{a}^\dagger \hat{a}\,|\, n\rangle = \hat{a}^\dagger \sqrt{n}\,|\, n-1\rangle \;\; = n\,|\, n\rangle$

$\hat{N} = \hat{a}^\dagger \hat{a}$ Informs you of what level the system is in

So $\hat{h}\,|\, n\rangle = \left[\left(\hat{a}^\dagger \hat{a}\right) + \frac{1}{2}\right]|\, n\rangle = \left(n + \frac{1}{2}\right)|\, n\rangle$

Recall $\hat{H} = \hbar\omega_0 \hat{h}$

So the energy levels go as $(n + ½)\hbar\omega_0$

Exactly what we found in solving the DE for the HO problem

Doing integrals

$$\left\langle n \mid \hat{\zeta} \mid n \right\rangle = ?$$

$$\hat{\zeta} = \frac{\hat{a}^{\dagger} + \hat{a}}{\sqrt{2}}$$

$$\langle n \mid \frac{\hat{a}^{\dagger} + \hat{a}}{\sqrt{2}} \mid n \rangle = \frac{1}{\sqrt{2}}\left(\left\langle n \mid n+1 \right\rangle \sqrt{n+1} + \left\langle n \mid n-1 \right\rangle \sqrt{n} \right) = 0$$

$$\left\langle n \mid \zeta \mid m \right\rangle = \frac{1}{\sqrt{2}}\left[\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1} \right]$$

$$\left\langle m \mid \hat{\zeta}^{2} \mid m \right\rangle =?$$

$$\hat{\zeta} = \frac{\hat{a}^{\dagger} + \hat{a}}{\sqrt{2}}$$

$$\hat{\zeta}^{2} = \frac{a^{\dagger} a^{\dagger} + \hat{a}\hat{a} + \hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}}{2}$$

Or we can make use of $I = \sum \left| n >< n \right|$

$$\left\langle m \mid \zeta^{2} \mid m \right\rangle = \sum_{n} \left\langle m \mid \zeta \mid n \right\rangle \left\langle n \mid \zeta \mid m \right\rangle$$

$$a^\dagger |0\rangle = |1\rangle,\ a^\dagger |1\rangle = \sqrt{2}\,|2\rangle,\ a^\dagger |2\rangle = \sqrt{3}\,|3\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger\right)^2 |0\rangle,\ |3\rangle = \frac{1}{\sqrt{6}}\left(a^\dagger\right)^3 |0\rangle$$

$$\phi_n(\zeta) = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}\phi_0(\zeta) = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\zeta - \frac{1}{\sqrt{2}}\frac{d}{d\zeta}\right)^n \phi_0(\zeta)$$

Can show that $\phi_0(\zeta) = A_0 e^{-\zeta^2/2}$

and eventually get to $\phi_n(\zeta) = \frac{1}{n!2^n\sqrt{\pi}} e^{-\zeta^2/2} H_n(\zeta)$

(different normalization constant if using variable x)

Often one finds the step-up and step-down operators defined directly in terms of x and y.

$$\hat{a}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p})$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^{\dagger} + \hat{a})$$

$$\hat{p} = \sqrt{\frac{\hbar m\omega}{2}}i(\hat{a}^{\dagger} - \hat{a})$$