Chapter 6 – Commutators

The values of two different observables, *a* and *b*, can be simultaneously determined (precisely) only if the measurement does not change the state of the system.

$$a \leftrightarrow \hat{A} \qquad\qquad b \leftrightarrow \hat{B}$$

$$\hat{B}\hat{A}\psi_n(x) = \hat{B}\alpha_n\psi_b(x), \quad \text{if } \ \psi_n \text{ an e.f. of } \hat{A} \qquad (A\psi_n = \alpha_n\psi_n)$$

$$= \beta_n\alpha_n\psi_n, \qquad \text{if } \psi_n \text{ also an e.f. of } \hat{B} \qquad \left(B\psi_n = \beta_n\psi_n\right)$$

Note that for this case $\hat{B}\hat{A}\psi_n = \hat{A}\hat{B}\psi_n$

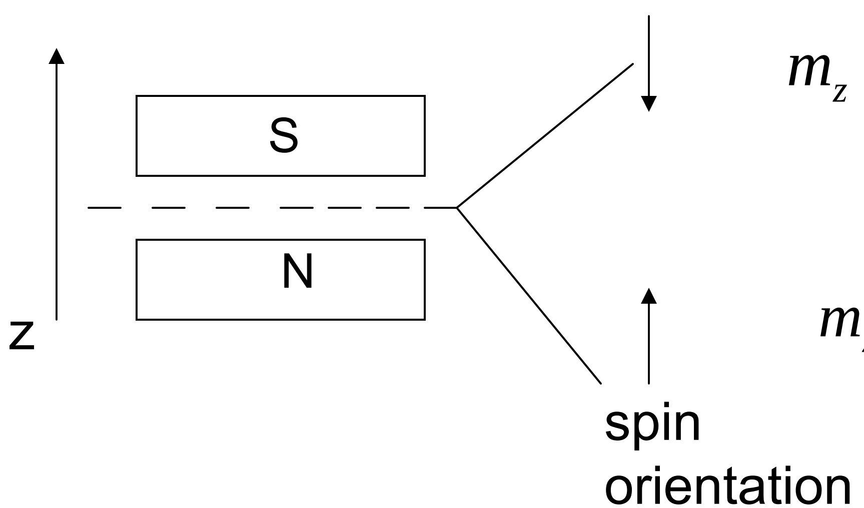
$$(\hat{A}\hat{B} - \hat{B}\hat{A})f = \underbrace{[A,B]}_{\text{commutator}}f$$

[A,B] = 0 $\Rightarrow$ **A and B commute**: the corresponding observables can be determined exactly, simultaneously

p_x, x cannot be known exactly

p_x, H cannot be known exactly if V ≠ constant)

Stern-Gerlach experiment



$$m_z = -\frac{\hbar}{2}$$

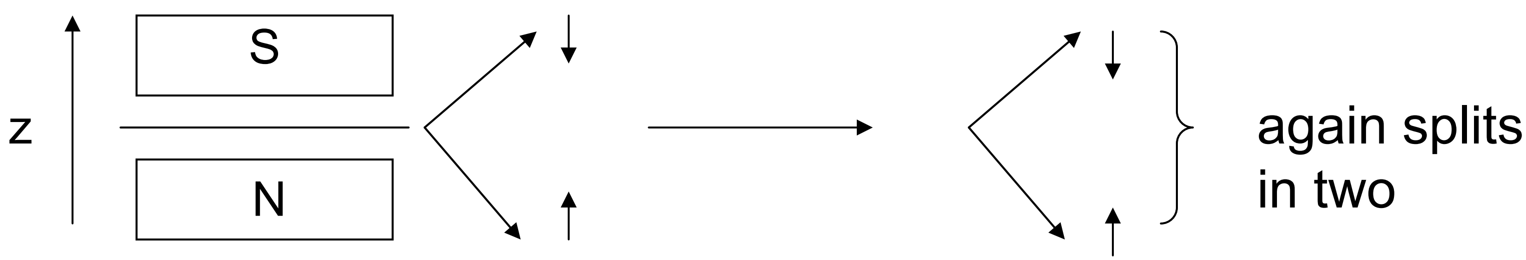
$$m_z = \frac{\hbar}{2}$$

Beam of Ag atoms split by inhomogeneous magnetic field

Ag atoms - unpaired e^- has spin
→ **magnetic moment** in z direction
→ deflected by external magnetic field

Expt. → only 2 values of spin possible in the z direction
→ eigenfunctions α (up spin), β (down spin)

Initial wave function $$\psi = \frac{1}{\sqrt{2}}(c_1\alpha + c_2\beta), \quad |c_1|^2 + |c_2|^2 = 1$$



run downward deflected
atoms through an inhomogeneous
mag. field in x direction

operator $\hat{A}$ $(= \hat{S}_z)$

(measures z component)

operator $\hat{B}$ $(= \hat{S}_x)$

(measures x components)

Now, take beam of the downward deflected atoms and pass through magnetic field in z direction

The beam is split in two (α, β) components

$\Rightarrow$ $\hat{A}$ and $\hat{B}$ do not commute

μ_z and μ_x cannot be simultaneously well defined

Stern-Gerlach expt. (1921) was carried out to confirm the Bohr model

Electron spin was discovered several years later.

Uncertainty principle (Heisenberg)

$$\Delta p \bullet \Delta x \geq \frac{\hbar}{2}$$

$\neq$ 0 because $\hat{p}_x$ and $\hat{x}$ do not commute

$$\sigma_\rho \sigma_x \geq \frac{\hbar}{2}$$

standard deviations

$$\sigma_x^2 = \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 \qquad \sigma_p^2 = \left\langle p^2 \right\rangle - \left\langle p \right\rangle^2$$

$$\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \langle x \rangle)^2} = \sqrt{\left\langle x^2 \right\rangle - \left\langle x \right\rangle^2}$$

spread in x

particle-in-box example

$$\langle x \rangle = \frac{a}{2}$$

$$\langle x^2 \rangle = a^2 \left(\frac{1}{3} - \frac{1}{2\pi^2 n^2} \right)$$

$$\langle p \rangle = 0$$

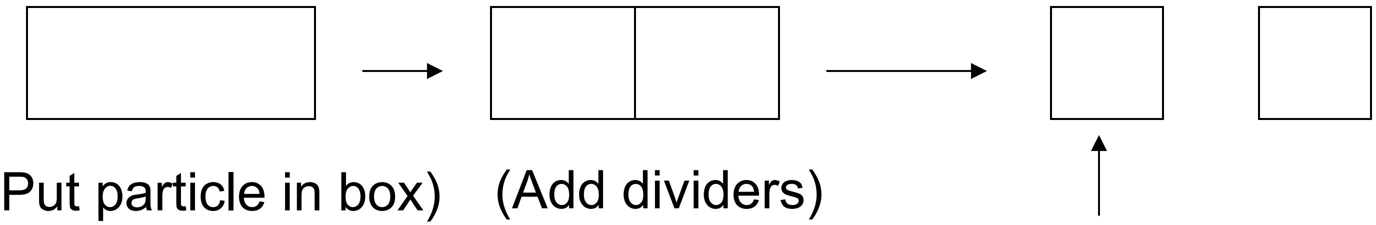
$$\langle p^2 \rangle = \frac{n^2 \pi^2 \hbar}{a^2}$$

$$\sigma_p = \frac{n\pi\hbar}{a}$$

$$\sigma_x = a\sqrt{\left(\frac{1}{12} - \frac{1}{2\pi^2 n^2} \right)}$$

$$\sigma_p \sigma_x = 0.57\hbar > \frac{\hbar}{2} \text{ for } n = 1$$

Supplemental material:



(Put particle in box) (Add dividers)

Open this box. Is the particle present?
(≡ apply position operator)

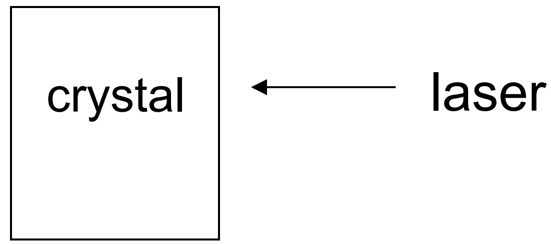
In one measurement either get ψ_{left} or ψ_{right}

$$\psi = a\psi_{left} + b\psi_{right} \qquad (|a|^2 + |b|^2 = 1, \ \text{assume } a = b)$$

Implications of looking into left-hand box

Entanglement, Teleportation, and Quantum Computers

$\psi = a\psi_{left} + b\psi_{right}$: superposition of a single particle



If we look in box and find particle there a^2 goes from ½ to 1 and b^2 drops from ½ to 0

- photons exit crystal with ½ original frequency
- one photon in → two photons out
  polarization can be horizontal (H) or vertical (V)

whatever polarization is measure for one photon,
the opposite is found for the other

entanglement $$\psi = \frac{1}{\sqrt{2}}\left(\psi_1(H)\psi_2(V) + \psi_1(V)\psi_2(H)\right)$$

Action at a distance

implications for teleportation and quantum computing