1

$$\int_0^{2\pi} e^{-im\theta}e^{in\theta}d\theta = \int_0^{2\pi} e^{i(n-m)\theta}d\theta = \frac{1}{i(n-m)}e^{i(n-m)\theta}]_0^{2\pi} = \frac{1}{i(n-m)}\left(e^{i(n-m)2\pi}-1\right)$$

Since (n-m) is an integer, the $e^{i(n-m)2\pi} = e^0 = 1$. Then the integral will be 1-1=0. Alternatively we can expand the Euler's formula:

$$\frac{1}{i(n-m)}\left(e^{i(n-m)2\pi}-1\right) = \frac{1}{i(n-m)}(\cos(n-m)2\pi + \mathrm{i}\sin(n-m)2\pi - 1) = \frac{1}{i(n-m)}(1+0-1)$$
$$= 0$$

* As a side note, while there are many ways to think of the meaning of $e^{i\theta}$, I've particularly found the explanation at this link (http://betterexplained.com/articles/intuitive-understanding-of-eulers-formula/) both intuitive and useful.

2

$$\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{2}{r}\right)Ae^{-br} = \frac{1}{r^2}\frac{d}{dr}r^2\left(-Abe^{-br}\right)+\frac{2}{r}Ae^{-br}$$
$$= \frac{1}{r^2}\left(-2rAbe^{-br}+r^2Ab^2e^{-br}\right)+\frac{2}{r}Ae^{-br} = -\frac{2}{r}Abe^{-br}+Ab^2e^{-br}+\frac{2}{r}Ae^{-br}$$

If b=1, then the function will be an eigen function of the operator (for any non-zero A).

3

$$1 = A^2\int_0^a\left(\frac{x}{a}\right)^4\left(1-\frac{x}{a}\right)^2dx = A^2\int_0^a\left(\frac{x}{a}\right)^4\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)dx = A^2\int_0^a\left(\frac{x^4}{a^4}-\frac{2x^5}{a^5}+\frac{x^6}{a^6}\right)dx$$
$$= A^2\left(\frac{x^5}{5a^4}-\frac{x^6}{3a^5}+\frac{x^7}{7a^6}\right)_0^a = A^2\left(\left(\frac{a}{5}-\frac{a}{3}+\frac{a}{7}\right)-0\right) = A^2\frac{a}{105}$$

$$1 = \frac{A^2a}{105} \Rightarrow A = \sqrt{\frac{105}{a}}$$

$$\langle x\rangle = \int_0^a\psi^*x\psi dx = \int_0^a x\left(\sqrt{\frac{105}{a}}\left(\frac{x}{a}\right)^2\left(1-\frac{x}{a}\right)\right)^2dx = \frac{105}{a}\int_0^a\left(\frac{x^5}{a^4}-\frac{2x^6}{a^5}+\frac{x^7}{a^6}\right)dx$$
$$= \frac{105}{a}\left(\frac{x^6}{6a^4}-\frac{2x^7}{7a^5}+\frac{x^8}{8a^6}\right)_0^a = \frac{105}{a}\left(\frac{a^2}{6}-\frac{2a^2}{7}+\frac{a^2}{8}\right) = \frac{105}{168}a = \frac{5}{8}a$$

$$\langle x^2\rangle = \int_0^a \psi^* x^2 \psi dx = \int_0^a x^2 \left(\sqrt{\frac{105}{a}}\left(\frac{x}{a}\right)^2\left(1-\frac{x}{a}\right)\right)^2 dx = \frac{105}{a}\int_0^a \left(\frac{x^6}{a^4} - \frac{2x^7}{a^5} + \frac{x^8}{a^6}\right) dx$$

$$= \frac{105}{a}\left(\frac{x^7}{7a^4} - \frac{2x^8}{8a^5} + \frac{x^9}{9a^6}\right)_0^a = \frac{105}{a}\left(\frac{a^3}{7} - \frac{2a^3}{8} + \frac{a^3}{9}\right) = \frac{105}{252}a^2 = \frac{5}{12}a^2$$

4

$$E = \frac{h^2}{8m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{a^2}\right) = \frac{h^2}{8m}\left(\frac{n_x^2 + n_y^2}{a^2}\right)$$

$\text{level 1 } (1,1) \Rightarrow \text{non degenerate} \Rightarrow \frac{h^2}{8m}\left(\frac{2}{a^2}\right)$

$\text{level 2 } (1,2 \ and \ 2,1) \Rightarrow \text{degeneracy} = 2 \Rightarrow \frac{h^2}{8m}\left(\frac{5}{a^2}\right)$

$\text{level 3 } (2,2) \Rightarrow \text{non degenerate} \Rightarrow \frac{h^2}{8m}\left(\frac{8}{a^2}\right)$

$\text{level 4 } (3,1 \ and \ 1,3) \Rightarrow \text{degeneracy} = 2 \Rightarrow \frac{h^2}{8m}\left(\frac{10}{a^2}\right)$

$\text{level 5 } (3,2 \ and \ 2,3) \Rightarrow \text{degeneracy} = 2 \Rightarrow \frac{h^2}{8m}\left(\frac{13}{a^2}\right)$

$\text{level 6 } (4,1 \ and \ 1,4) \Rightarrow \text{degeneracy} = 2 \Rightarrow \frac{h^2}{8m}\left(\frac{17}{a^2}\right)$

Note that the n_x and n_y values are shown in parenthesis. Also based on E, the sixth level is (4,1) and not (3,3).

5

$$\psi_{n_x n_y} = \left(\sqrt{\frac{2}{a}}\sin\frac{n_x\pi x}{a}\right)\left(\sqrt{\frac{2}{a}}\sin\frac{n_y\pi y}{a}\right) = \frac{2}{a}\sin\frac{n_x\pi x}{a}\sin\frac{n_y\pi y}{a}$$

To make the integration easier, we assign $\frac{n_x\pi}{a} = U \ and \ \frac{n_y\pi}{a} = V$. Also note that: $\int \sin^2 kx \, dx = \frac{x}{2} - \left(\frac{\sin 2kx}{4k}\right)$

$$P = \int \psi * \psi = \frac{4}{a^2} \int_{\frac{2a}{3}}^{a} \sin^2 Vy\, dy \int_{0}^{\frac{a}{3}} \sin^2 Ux\, dx \ = \frac{4}{a^2} \left(\frac{x}{2} - \left(\frac{\sin 2Ux}{4U} \right) \right)_0^{\frac{a}{3}} \left(\frac{y}{2} - \left(\frac{\sin 2Vy}{4V} \right) \right)_{\frac{2a}{3}}^{a}$$

$$\left(\frac{x}{2} - \left(\frac{\sin 2Ux}{4U} \right) \right)_0^{\frac{a}{3}} = \left(\frac{a}{6} - \frac{\sin 2 \frac{n_x \pi}{a} \frac{a}{3}}{4 \frac{n_x \pi}{a}} \right) - 0 = \frac{a}{6} - \frac{a \sin \frac{2n_x \pi}{3}}{4 n_x \pi}$$

$$\left(\frac{y}{2} - \left(\frac{\sin 2Vy}{4V} \right) \right)_{\frac{2a}{3}}^{a} = \left(\frac{a}{2} - \frac{\sin 2 \frac{n_y \pi}{a} a}{4 \frac{n_y \pi}{a}} \right) - \left(\frac{a}{3} - \frac{\sin 2 \frac{n_y \pi}{a} \frac{2a}{3}}{4 \frac{n_y \pi}{a}} \right)$$

$$= \left(\frac{a}{2} - \frac{a \sin 2 n_y \pi}{4 n_y \pi} \right) - \left(\frac{a}{3} - \frac{a \sin 4 \frac{n_y \pi}{3}}{4 n_y \pi} \right) = \left(\frac{a}{2} - 0 \right) - \left(\frac{a}{3} - \frac{a \sin 4 \frac{n_y \pi}{3}}{4 n_y \pi} \right)$$

$$= \frac{a}{6} + \frac{a \sin 4 \frac{n_y \pi}{3}}{4 n_y \pi}$$

For the ground state where the nx=ny=1, the P will be:

$$P = \frac{4}{a^2} \left(\frac{a}{6} - \frac{a \sin \frac{2\pi}{3}}{4\pi} \right) \left(\frac{a}{6} + \frac{a \sin \frac{4\pi}{3}}{4\pi} \right) = \frac{4}{a^2} \left(\frac{a}{6} - \frac{a\sqrt{3}}{8\pi} \right)^2 = 4 \left(\frac{1}{6} - \frac{\sqrt{3}}{8 * 3.14} \right)^2 = 0.03819$$