# Chapter 6 – Commutators

The values of two different observables, *a* and *b*, can be simultaneously determined (precisely) only if the measurement does not change the state of the system.

$$a \leftrightarrow \hat{A} \qquad\qquad b \leftrightarrow \hat{B}$$

$$\hat{B}\hat{A}\psi_n(x) = \hat{B}\alpha_n\psi_n(x), \quad \text{if } \psi_n \text{ an e.f. of } \hat{A} \quad (A\psi_n = \alpha_n\psi_n)$$

$$= \beta_n\alpha_n\psi_n, \quad \text{if } \psi_n \text{ also an e.f. of } \hat{B} \quad \left(B\psi_n = \beta_n\psi_n\right)$$

Note that for this case $\hat{B}\hat{A}\psi_n = \hat{A}\hat{B}\psi_n$

$$(\hat{A}\hat{B} - \hat{B}\hat{A})f = \underbrace{[A, B]}_{\text{commutator}} f$$

commutator

$[A,B] = 0 \Rightarrow$ **A and B commute**: the corresponding observables can be determined exactly, simultaneously

$p_x$, x cannot be known exactly

$p_x$, H cannot be known exactly if V depends on x)

Consider the particle in the box problem

Does V depend on x?

If one measures the energy and gets $h^2/8ma^2$, the system is in the ground eigenstate of H.

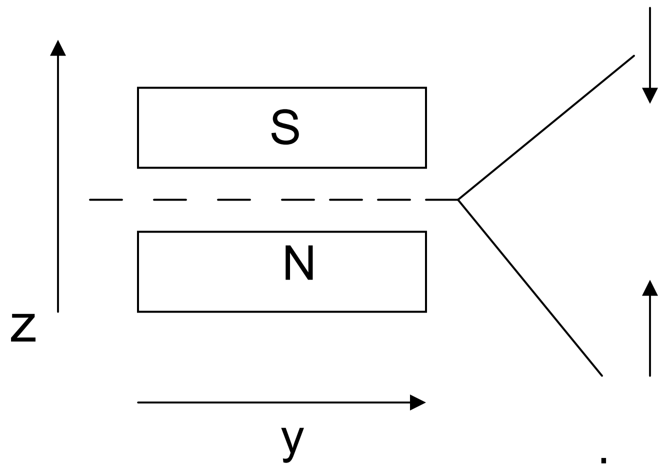
What if you now measure the momentum what do you get?

What is the average momentum?

What happens if you measure the momentum and then measure the energy?

# Stern-Gerlach experiment

Beam of Ag atoms moving in y direction, in an inhomogeneous magnetic field in the z direction




$$m_z = -\frac{\hbar}{2}$$

$$m_z = \frac{\hbar}{2}$$

spin orientation

Beam of Ag atoms split by inhomogeneous magnetic field – 2 well-separated peaks seen

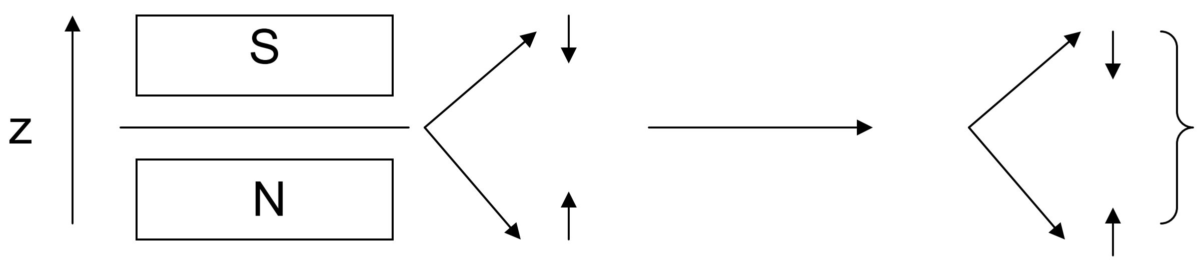
Ag atoms have an unpaired electron, which means that Ag atoms have magnetic moments (will consider this later).

Classically expect there to be a continuous distribution of Ag atoms along the z-direction, as we would expect the z component of the magnetic moment to be able to assume any value between the two extremes.

Ag atoms - unpaired $e^-$ has spin
→ **magnetic moment** in z direction
→ deflected by external magnetic field

Expt. → only 2 values of spin possible in the z direction
→ eigenfunctions $\alpha$ (up spin), $\beta$ (down spin)

Initial wave function $\psi = (c_1\alpha + c_2\beta), \quad |c_1|^2 + |c_2|^2 = 1$




again splits in two (but now along the x direction)

run downward deflected atoms through an inhomogeneous mag. field in x direction

operator $\hat{A}$ $(= \hat{S}_z)$

(measures z component)

operator $\hat{B}$ $(= \hat{S}_x)$

(measures x component)

Now, take beam of the downward deflected atoms and pass through magnetic field in z direction

The beam is split in two ($\alpha$, $\beta$) components

$\Rightarrow$ $\hat{A}$ and $\hat{B}$ do not commute

i.e., $\mu_z$ and $\mu_x$ cannot be simultaneously well defined

Stern-Gerlach expt. (1921) was carried out to confirm the Bohr model

They actually thought it confirmed the Bohr model.

Electron spin was discovered several years later.

See: http://lorentz.leidenuniv.nl/history/spin/goudsmit.html

# Uncertainty principle (Heisenberg)

$$\Delta p \bullet \Delta x \geq \frac{\hbar}{2}$$

$\neq$ 0 because $\hat{p}_x$ and $\hat{x}$ do not commute

$$\sigma_\rho \sigma_x \geq \frac{\hbar}{2}$$

standard deviations

$$\sigma_x^2 = \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 \qquad \sigma_p^2 = \left\langle p^2 \right\rangle - \left\langle p \right\rangle^2$$

$$\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \langle x \rangle)^2} = \sqrt{\left\langle x^2 \right\rangle - \left\langle x \right\rangle^2}$$

spread in x

particle-in-box example

$$\langle x \rangle = \frac{a}{2}$$

$$\langle x^2 \rangle = a^2\left(\frac{1}{3} - \frac{1}{2\pi^2 n^2}\right)$$

$$\langle p \rangle = 0$$

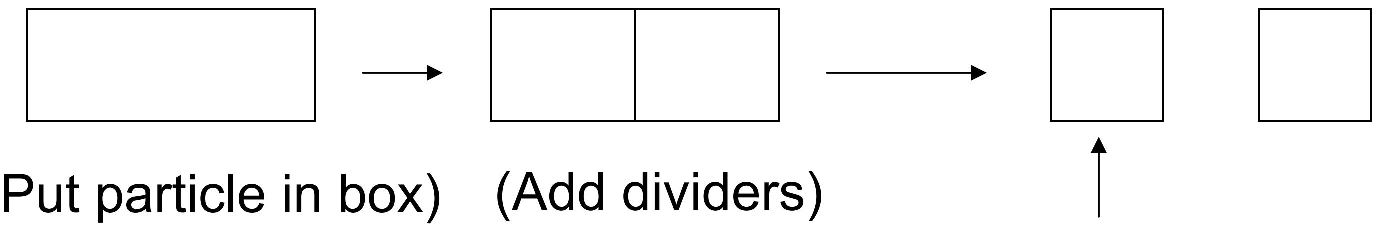
$$\langle p^2 \rangle = \frac{n^2\pi^2\hbar}{a^2}$$

$$\sigma_p = \frac{n\pi\hbar}{a}$$

$$\sigma_x = a\sqrt{\left(\frac{1}{12} - \frac{1}{2\pi^2 n^2}\right)}$$

$$\sigma_p\sigma_x = 0.57\hbar > \frac{\hbar}{2} \text{ for } n = 1$$

Supplemental material:



(Put particle in box) (Add dividers)

Open this box.
Is the particle present?
(≡ apply position operator)

In one measurement either get $\psi_{left}$ or $\psi_{right}$

$$\psi = a\psi_{left} + b\psi_{right} \qquad (|a|^2 + |b|^2 = 1, \ \text{assume } a = b)$$

Implications of looking into left-hand box

If we find particle there, then a=1, and b =0. The right-hand box instantaneously "knows" what happened in the left-hand box, no matter how far away it is. (Action at a distance)

A system with no net magnetic moment decays to particles with magnetic moments ($m_z$) $\pm 1/2$

If one particle is +1/2, the other must be -1/2 since net moment is $\phi$

Two possibilities: $\underset{1}{\uparrow} \quad \underset{2}{\downarrow}$ or $\underset{1}{\downarrow} \quad \underset{2}{\uparrow}$ $\longleftarrow$ particle label

So $\psi = \frac{1}{\sqrt{2}}\left(\uparrow\downarrow + \downarrow\uparrow\right)$

Neither particle has well defined magnetic moment value until a measurement is made.

Two entangled photons sent through two optical fibers 10 km apart.

If one is sent through a slit causing diffraction, the second photon displays a diffraction pattern even though it was not sent through a slit.

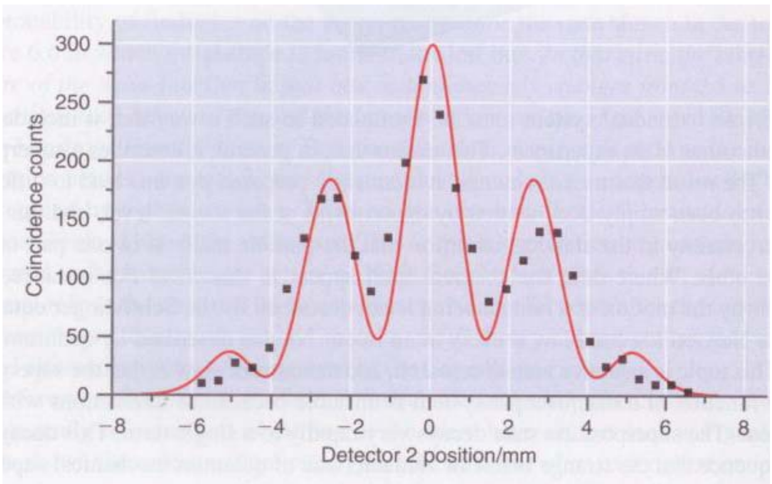



Figure from Engel, Original data from Phys. Rev. Lett. 74, 3600 (1995)

Can information be transmitted over a distance for a quantum system?

Suppose $\psi = \sum_m b_m \phi_m$

↑ eigenfunctions of some operator

measurements give $|b_m|^2$ values

phase information is lost
so cannot copy system

sign information lost
can even be more
complicated, e.g.,

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

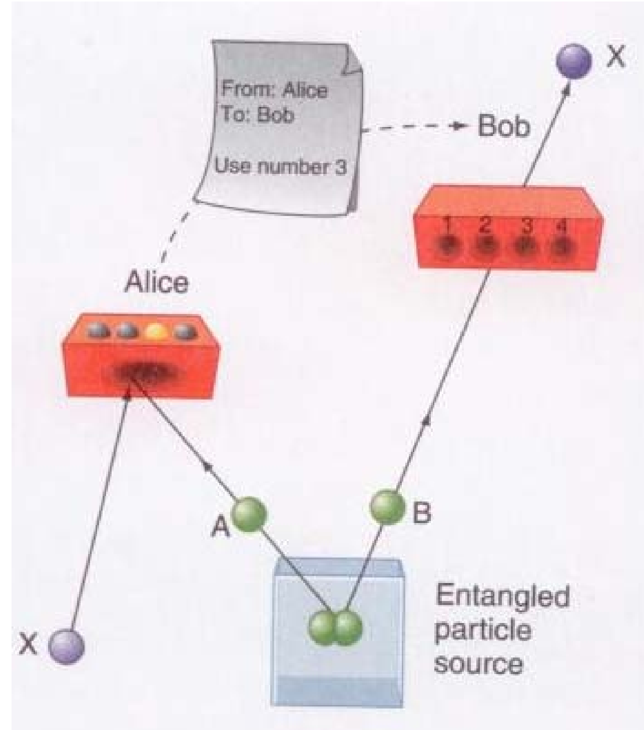
In 2000, experiments were done in which a atom at one position was recreated at another position.

Bob receives photon *B* and
Alice receives photon *A*.

Each stores his/her photon so
entanglement is preserved.

Now later, Alice wants to send photon *X* to Bob.

She cannot measure polarization of *X*, as
that would change *X*'s wavefunction.




She entangles *X* and *A*.

Whatever state *X* is in, *A* is in the other (orthogonal state)
(If *X* is vertical pol., *A* is horizontal and *visa versa*)
Whatever state *A* has, *B* is in orthogonal state.

If $B \perp A$ and $A \perp X$, *B* and *X* must be in the same state

*B* acquires polarization of *X*.

Alice had to entangle *A* and *X*, so properties of *X* changed at her location and transferred to Bob's location.

At Bob's location B acquires the polarization that Alice's X originally had

Photon has been teleported but not copied.

Neither Bob nor Alice knows state of *X*.

The actual situation is actually more complicated than described above

Alice's entanglement of *A + X* has four outcomes. She can measure after the entanglement which one she has.

Bob has four possible outcomes.

Alice must tell Bob which entanglement she has, so Bob can know how to rotate *B* to make identical to *X*.

Because of this, the limit is the speed of light.

Recently, the experiment has been carried out with entangled atoms.

## Quantum computing

classical computer – info stored in bits (0, 1)

*n* bits → $2^n$ states

quantum computers:  qubits – simultaneously a combination of *0* and 1

3 bits → 8 #'s

By preparing entangled states can store more than one number simultaneously.

Thus doing operations with entangled qubits can allow huge numbers of calculations to be done in parallel