Distorted Risk Measures with Application to Military Capability Shortfalls[1]

Edwin J. Offutt, Jeffrey P. Kharoufeh[2], Richard F. Deckro

Department of Operational Sciences
Air Force Institute of Technology
2950 Hobson Way (AFIT/ENS)
Wright Patterson AFB OH 45433-7765 USA
Voice: (937) 255-3636 x4603;
Fax: (937) 656-4943;
Email: Jeffrey.Kharoufeh@afit.edu

Abstract

We consider the problem of selecting an appropriate distortion function and associated parameters to account for rare but catastrophic events that may result from a shortfall of military or security capabilities. Additionally, we describe the means by which a decision maker may allocate resources among various risk-mitigating systems, subject to a finite budget constraint, while considering the risk of such shortfalls. Through a numerical illustration, we show that the optimal allocation of resources is sensitive to the decision maker's level of risk aversion.

Keywords: Distortion function, risk measures, capability.

[1]The views expressed in this paper are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government. Pre-publication draft. Do not quote without the authors' permission.

[2]Corresponding author.

1 Introduction

In this paper, we review and illustrate the concept of *distorted* risk measures in order to analyze the risk of shortfalls in military capabilities. The application of distorted risk measures facilitates a method for optimally allocating resources among various military capabilities while taking into account the risk of capability shortfalls. However, an open and difficult question is the selection of appropriate distortion functions (and parameters), and their interpretations, for a given risk scenario. In this study, we propose numerical measures that can be used to assist in distortion function selection. Moreover, it is our aim to elucidate the usefulness of distorted risk measures, especially for scenarios involving low-likelihood, catastrophic events.

Consider a military or national-level decision maker who is faced with addressing shortfalls in military or homeland security capability. However, due to budgetary (or possibly other) constraints, only a subset of shortfalls can be addressed. We assume that input data from subject matter experts, in the form of risks of capability shortfalls, can be converted into appropriate risk distributions using, for example, techniques such as those outlined in Clemen and Reilly [4] for assessing continuous or discrete probabilities. While the decision maker may trust and value the opinions of subject matter experts, he or she may desire to assign their own risk priorities in the resource allocation process to reflect additional information and/or considerations not necessarily available to the subject matter experts.

Distortion functions can be used to alter standard risk measures for scenarios in which low-likelihood, yet potentially catastrophic, occurrences in the tail of the risk distribution are of interest but are often suppressed by standard risk measures (e.g., expectation and conditional expectation). In such cases, distortion functions serve the purpose of shifting probability density toward the region of the distribution that corresponds to highly adverse outcomes, thereby inflating the expectation risk measure. A wide variety of shaping effects and degrees of effect are possible depending on the distortion function selected and its parameters. The challenge for the decision maker is to select appropriate distortion functions to apply to risk distributions suggested by his or her subordinates. Similarly, because the degree of distortion applied via the parameter selection can be (directly or indirectly) linked to the decision maker's degree of risk aversion, the selection of appropriate distortion function parameters must also be considered. While numerous distortion functions have been introduced in the mathematical finance and insurance literature (cf. [7] and [11]), there is not a universally accepted, formal methodology for the selection of a distortion function or its associated parameters.

As a relatively new and competing theory for the pricing of risk (prospect theory is the other), the properties of parametric distortion functions have been examined in the finance and insurance literature for the past ten years. The seminal work on distortion functions is due to Wang [10], who first proposed transforming the survivor function of the risk using the proportional hazard transform. Subsequently, Wang [11] generalized the theory of distortion to an entire class of functions used to calculate insurance premiums. Wang *et al.* [13] provided an axiomatic theory of insurance premiums pricing. Distortion and the axiomatic theory are closely tied to the concept of risk measure *coherency*, outlined by Artzner *et al.* [1] and further developed by the same authors in [2]. Simply stated, a coherent risk measure is one that accurately portrays the way financial markets operate. Artzner *et al.* [1] establish four attributes that a coherent risk measure must possess, the most important being that of subadditivity which means that the aggregation of several risks should not increase the overall risk. McLeish and Reesor [7] proved that a concave distortion function produces a coherent risk measure. Wirch and Hardy [14] made two general observations regarding distortion parameters. First, they associate the parameters with a decision maker's risk aversion level toward risk in the far right tail of the distribution. Second, they state that the selection of distortion parameters is mostly a "political" decision. To our knowledge, the problem of selecting an appropriate distortion function and its parameter(s) has not been formally addressed in the risk analysis literature.

In this paper we seek to provide some guidance for the selection of distortion functions and their parameters when concerned with inflating the right tail of a risk distribution. We are motivated by the potentially catastrophic losses that may result from military or homeland security capability shortfalls. Specifically, this concerns the representation of catastrophic risks that cannot, for operational or political reasons, be disregarded despite their low likelihood of occurrence. Throughout this study, the focus of our attention is on the expected value of the risk; however, it is worth noting that we may choose any other coherent risk measure. Though it is admittedly difficult to generalize the guidelines to an arbitrary scenario, we attempt to provide a framework within which the risk of military capability shortfalls may be considered. For other contexts (e.g., insurance or financial risk), it may be necessary to vary the framework or even develop a separate analysis.

We summarize and study the impact of three of the most widely referenced distortion functions on four parametric probability distributions, specifically the exponential, Weibull, triangular, and uniform distributions. Whenever possible, we provide closed-form expressions for the corresponding distorted risk measures. However, when such expressions are not available, it is still possible to compute the measures by numerical methods. We propose two simple measures of distortion effects, *effectiveness* and *efficiency*, and by means of a simple designed experiment, we argue that some distortions may be preferable to others, depending on the risk distribution and the extent to which distortion is desired. Finally, we illustrate the means by which a decision maker's risk aversion levels may be incorporated into a resource allocation problem using appropriate distortion selection. The results of the study offer some practical guidance for the application of distortion functions to some specific risk scenarios.

The remainder of this paper is organized as follows. In section 2, we define and review the concept of distorted risk measures and provide analytical results for the distorted expectation risk measure using a few parametric probability distributions. In section 3 we propose two distortion performance measures and use a simple designed experiment to help establish some guidelines for distortion function selection. Section 4 presents an illustrative example of a resource allocation problem which considers the decision maker's risk aversion levels while section 5 provides a few concluding remarks and future research directions.

2 A Review of Distortion Functions

This section provides a brief overview of distortion functions and coherent risk measures. Before presenting mathematical descriptions, we first provide an intuitive motivation for the use of distortions in a given risk scenario.

2.1 Concept of Distortion

Assume that risk is a nonnegative random variable. If one is concerned only about the probability that the random variable is above (or below) some critical value, and not about what happens above that value, then it is instructive to use a simple quantile risk measure (i.e., the Value-at-Risk (VaR) measure in finance). In such a case, the distortion function is simply a step function and the resulting distorted risk measure has no probability in the tail of the original risk distribution. In many cases, however, the tail of the distribution is of interest because unacceptable, highly catastrophic losses can occur with low probability. In these scenarios, it makes sense to amplify the probability in the region of the original risk distribution that corresponds to highly adverse and unacceptable outcomes. In our context, a military or national-level decision maker may forego the opportunity to acquire certain military capabilities. The risks involved in a shortfall of military capability can be viewed as the potential for loss of human life, loss of assets, or other significant losses.

We assume that a nonnegative risk X is defined on an appropriate probability space $(\Omega, \mathcal{F}, P)$ with cumulative distribution function (c.d.f.) given by $F(x) = P(X \leq x)$, $x \geq 0$ and survivor function $S(x) \equiv P(X > x)$, $x \geq 0$. The expectation risk measure is given by

$$E(X) = \int_0^\infty S(x)dx. \tag{1}$$

The problem of selecting the risk distribution for extreme events is a difficult question in its own right. We do not specifically address distribution selection here; however, some guidance is given by Lambert *et al.* [5]. The objective of a distortion function is to transform the survivor function $S(x)$ so that when a risk measure is computed, the resulting distorted measure more adequately reflects the possibility and impact of extreme events. More formally, the distortion of S is given by the composition function

$$g(S(x)) \equiv (g \circ S)(x), \tag{2}$$

where g is a function satisfying (cf. [14]):

1. $g : [0,1] \to [0,1]$ is monotonically increasing;

2. $\lim_{u \downarrow 0} g(u) = 0$; and

3. $\lim_{u \uparrow 1} g(u) = 1$.

The function $\hat{S}(x) \equiv (g \circ S)(x)$ is again a survivor function with the usual properties: its range is $[0,1]$, it is non-increasing in x, and integrating over the range of X gives the (distorted) expectation. That is, under the distortion g, the expectation risk measure is

$$\hat{E}(X) = \int_0^\infty \hat{S}(x)dx = \int_0^\infty (g \circ S)(x)dx. \tag{3}$$

The concept of distortion is closely tied to the concept of risk measure coherency which was formalized by Artzner *et al.* [2]. Suppose X and Y are two nonnegative random variables representing two risks and let ρ denote a risk measure. Then ρ is said to be a coherent risk measure if it satisfies the following four axioms:

1. Translation invariance: For all real α and r, $\rho(X + \alpha r) = \rho(X) - \alpha$

2. Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$

3. Positive homogeneity: For all $\alpha \geq 0$, $\rho(\alpha X) = \alpha \rho(X)$

4. Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$

McLeish and Reesor [7] have shown that, if g is a concave function, then the resulting distorted risk measure will satisfy the four axioms of coherency. This fact will be useful in determining appropriate distortion parameter values since we seek a resulting risk measure that is coherent. Next, we review some of the most commonly applied distortions.

2.2 Common Distortion Functions

The distortion functions most frequently encountered in the literature are the gamma-beta distortion and its variants which are discussed extensively in [7]. This family of distortion functions consists of the gamma-beta, beta, proportional hazard (PH), dual power (DP), gamma, and exponential (EX) distortions. Among these six, the single-parameter distortions (PH, DP, and EX) will be considered for three primary reasons: i) the effects of an individual parameter may be observed more easily; ii) the distorted expectation risk measure can be computed analytically in many cases, and numerically in others; and iii) it is desirable to minimize the number of parameters that need to be estimated.

The gamma-beta distortion is defined as

$$g_{GB}(S(x)) = \int_0^{S(x)} K t^{a-1}(1-t)^{b-1} \exp(-t/c)dt, \tag{4}$$

where

$$K^{-1} = \int_0^1 t^{a-1}(1-t)^{b-1} \exp(-t/c)dt.$$

This distortion serves as the basis for other distortions when we assume certain values for the parameters a, b, and c. It is worth mentioning here that a, b, and c may assume any nonnegative values; however, McLeish and Reesor [7] have shown that $0 \le a \le 1$, $b \ge 1$, and $c \ge 0$ are sufficient to ensure concavity of the distortion function, and thus coherency of the associated risk measure.

By setting $b = 1$ and allowing $c \to \infty$ in (4), we obtain the proportional hazard (PH) distortion given by

$$(g_{PH} \circ S)(x) = S^a(x), \quad 0 \le a \le 1. \tag{5}$$

The attractive feature of (5) is its ease of computation. By setting $a = 1$ and allowing $c \to \infty$ in (4), we arrive at the dual power (DP) distortion given by

$$(g_{DP} \circ S)(x) = 1 - (1 - S(x))^b, \quad b \geq 1. \tag{6}$$

As noted by Wirch and Hardy [14], this distortion has perhaps the most lucid interpretation. For an integer value of b, the expectation risk measure corresponds to the expected value of the maximum of a sample of b observations of X. Finally, the exponential (EX) distortion depends only on the single parameter c and is given by

$$(g_{EX} \circ S)(x) = \frac{1 - e^{-S(x)/c}}{1 - e^{-1/c}}, \quad c \geq 0. \tag{7}$$

This distortion corresponds to an exponential random variable restricted to the interval $[0, 1]$.

The question with which we concern ourselves in this study is, "Why may one of the distortion functions be preferable to the others in a given context?" The answer to this question is that it depends upon the risk scenario under consideration. In Figure 1, each of the single-parameter distortions is applied to an exponentially distributed risk X with rate parameter $\lambda = 3.5$. The gamma-beta distortion is also included since it uses all three parameters a, b, and c. The undistorted exponential density is depicted by the solid line. Among the single-parameter distortions, the proportional hazard (PH) distortion has the greatest effect on the right tail of the distribution, thickening it considerably. The dual power (DP) distortion, while inflating the right tail slightly, has a much more noticeable effect on the left side of the distribution, shifting the mode away from zero. The exponential (EX) distortion can best be described as a combination of the effects of the PH and DP. In general, these effects are consistent for the other distributions considered in this paper.

From this plot, we can clearly observe why a particular distortion function might be preferred over another. One may be more concerned with inflating the right tail rather than altering the left-hand side of the distribution. In what follows, we compute the distorted expectation risk measure using the three single-parameter distortion functions and four parametric probability distributions. These will be used to study the effects of distortions and

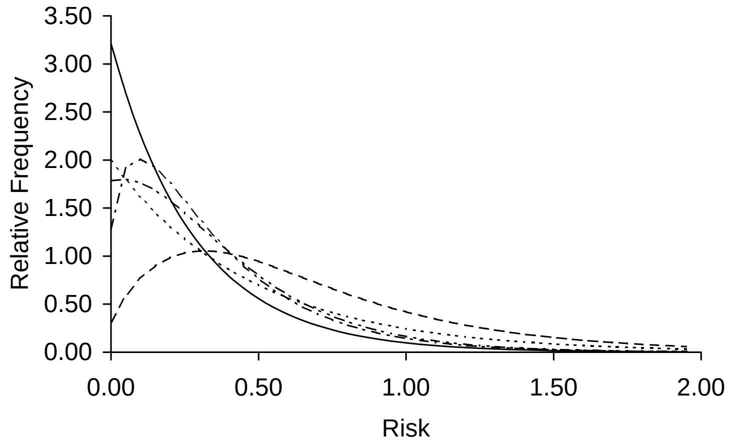

Figure 1: Distorted density when $X \sim \text{Exp}(3.5)$ with distortion parameters $a = 0.6$, $b = 1.5$, and $c = 0.8$ for GB (solid is no distortion, $- - -$ GB, $\cdots$ PH, $- \cdot -\cdot$ DP, $- \cdot\cdot -$ EX) .

their parameters in section 3.

2.3 Computing Distorted Measures

In order to elucidate the effects of distortion, we now apply the single-parameter distortions to a set of parametric probability distributions. The four distributions we consider are: exponential with rate parameter λ (denoted $\text{Exp}(\lambda)$); Weibull with shape parameter β and scale parameter θ (denoted $\text{Weib}(\beta, \theta)$); triangular on the closed interval $[\theta_1, \theta_2]$ with mode m (denoted $\text{Tria}(\theta_1, \theta_2, m)$); and continuous uniform on the closed interval $[\theta_1, \theta_2]$ (denoted $U(\theta_1, \theta_2)$). We select these four distributions because they are representative of risk distributions from a variety of disciplines including actuarial science, financial and insurance risk, as well as reliability. Moreover, they span a range of distribution shapes on both bounded and unbounded intervals. Finally, they have a relatively small number of parameters that may be easily estimated using information that is likely to be available from subject matter experts.

For each combination of distribution and distortion, we have attempted to summarize closed-form expressions for the distorted expectation risk measure given by equation (3). In some cases, explicit expressions are attainable, while others remain as definite or indefinite integrals that can be evaluated numerically using standard methods. These results are recorded in Tables 1 through 4.

First suppose the risk X is exponentially distributed with rate parameter $\lambda > 0$. In such case, the survivor function is given by

$$S(x) = \begin{cases} e^{-\lambda x} & \text{if } x \geq 0,\, \lambda > 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

The undistorted risk measure is $\mu_0 \equiv E(X) = \lambda^{-1}$. Table 1 provides a summary of the distorted survivor function and the distorted risk measure computed by equation (3).

Table 1: Distorted risk measures when $X \sim \text{Exp}(\lambda)$.

Distortion function	$\hat{S}(x)$	$\hat{E}[X]$
g_{PH}	$e^{-\lambda a x}$	$(\lambda a)^{-1}$
g_{DP}	$1-(1-e^{-\lambda x})^b$	$\int_0^\infty [1-(1-e^{-\lambda x})^b]\,dx$
g_{EX}	$\frac{1-\exp(-e^{-\lambda x}/c)}{1-\exp(-1/c)}$	$\int_0^\infty \frac{1-\exp(-e^{-\lambda x}/c)}{1-\exp(-1/c)}\,dx$

Next, suppose the risk follows a Weibull distribution with parameters β and θ. In such case, the survivor function is

$$S(x) = \begin{cases} \exp((-x/\theta)^{\beta}) & \text{if } x \geq 0,\, \beta > 0,\, \theta > 0 \\ 0 & \text{otherwise} \end{cases}, \tag{9}$$

and the undistorted expectation is $\mu_0 = (\theta/\beta)\Gamma(\beta^{-1})$, where $\Gamma(\cdot)$ is the gamma function. Similarly, the distorted risk measures are summarized in Table 2.

Table 2: Distorted risk measures when $X \sim \text{Weib}(\beta, \theta)$.

Distortion function	$\hat{S}(x)$	$\hat{E}[X]$
g_{PH}	$e^{a(-x/\theta)^{\beta}}$	$\frac{\theta}{\beta \sqrt[\beta]{a}} \Gamma\left(\frac{1}{\beta}\right)$
g_{DP}	$1 - (1 - e^{(-x/\theta)^{\beta}})^{b}$	$\int_0^{\infty} [1 - (1 - e^{(-x/\theta)^{\beta}})^{b}]\, dx$
g_{EX}	$\frac{1-\exp(-e^{(-x/\theta)^{\beta}}/c)}{1-\exp(-1/c)}$	$\int_0^{\infty} \frac{1-\exp(-e^{(-x/\theta)^{\beta}}/c)}{1-\exp(-1/c)}\, dx$

The third distribution we considered was the triangular distribution on $[\theta_1, \theta_2]$ with mode value m. The survivor function is given by

$$S(x) = \begin{cases} 1 & \text{if } x < \theta_1 \\ 1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)} & \text{if } \theta_1 \le x \le m \\ \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)} & \text{if } m < x \le \theta_2 \\ 0 & \text{if } x > \theta_2 \end{cases}, \tag{10}$$

where $\theta_1 < \theta_2$, $\theta_1 \le x \le \theta_2$, and $\theta_1 \le m \le \theta_2$. The undistorted expectation is $\mu_0 = (\theta_1 + \theta_2 + m)/3$.

Table 3: Distorted risk measures when $X \sim \text{Tria}(\theta_1, \theta_2, m)$.

Distortion function	$\hat{S}(x)$	$\hat{E}[X]$
g_{PH}	$\left(1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^a, \ \theta_1 \leq x \leq m$ $\left(\frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^a, \ m < x \leq \theta_2$	$\int_{\theta_1}^{m} \left(1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^a dx$ $+ \ \frac{(\theta_2-m)^{a+1}}{(2a+1)(\theta_2-\theta_1)^a}$
g_{DP}	$1 - \left(\frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^b, \ \theta_1 \leq x \leq m$ $1 - \left(1 - \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^b, \ m < x \leq \theta_2$	$m - \theta_1 - \frac{(m-\theta_1)^{b+1}}{(\theta_2-\theta_1)^b(2b+1)} \ +$ $\int_{m}^{\theta_2} \left[1 - \left(1 - \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^b\right] dx$
g_{EX}	$\frac{1-\exp\left(\frac{-1}{c} + \frac{(x-\theta_1)^2}{c(\theta_2-\theta_1)(m-\theta_1)}\right)}{1-\exp(-1/c)}, \ \theta_1 \leq x \leq m$ $\frac{1-\exp\left(\frac{-(\theta_2-x)^2}{c(\theta_2-\theta_1)(\theta_2-m)}\right)}{1-\exp(-1/c)}, \ m < x \leq \theta_2$	$\int_{\theta_1}^{m} \frac{1-\exp\left(\frac{-1}{c} + \frac{(x-\theta_1)^2}{c(\theta_2-\theta_1)(m-\theta_1)}\right)}{1-\exp(-1/c)} dx$ $+ \ \int_{m}^{\theta_2} \frac{1-\exp\left(\frac{-(\theta_2-x)^2}{c(\theta_2-\theta_1)(\theta_2-m)}\right)}{1-\exp(-1/c)} dx$

Finally, when the risk X is distributed $U(\theta_1, \theta_2)$, the survivor function is given by

$$S(x) = \begin{cases} 1 - \frac{x-\theta_1}{\theta_2-\theta_1} & \text{if } \theta_1 \leq x \leq \theta_2 \\ 0 & \text{otherwise} \end{cases}, \tag{11}$$

and the undistorted expectation is $\mu_0 = (\theta_1 + \theta_2)/2$. Table 4 summarizes the distorted survivor function and risk measure.

Table 4: Distorted risk measures when $X \sim U(\theta_1, \theta_2)$.

Distortion function	$\hat{S}(x)$	$\hat{E}[X]$
g_{PH}	$\left(1 - \frac{x-\theta_1}{\theta_2-\theta_1}\right)^a$	$(\theta_2 - \theta_1)\left(\frac{1}{a+1}\right)$
g_{DP}	$1 - \left(\frac{x-\theta_1}{\theta_2-\theta_1}\right)^b$	$(\theta_2 - \theta_1)\left(\frac{b}{b+1}\right)$
g_{EX}	$\frac{1-\exp(-(1-\frac{x-\theta_1}{\theta_2-\theta_1})/c)}{1-\exp(-1/c)}$	$(\theta_2 - \theta_1)\left(\frac{1-c+ce^{-1/c}}{1-e^{-1/c}}\right)$

It is important to note that, for the intractable results in Tables 1-3, the distorted risk measure may be approximated using numerical quadrature routines widely available in standard computing environments. In section 3, we propose measures that may be used to assess the effect of distortion and present a designed experiment to assist in establishing guidelines for appropriate distortion function selection.

3 Measuring Distortion Effects

In this section, we propose measures of effectiveness and efficiency to assist in selecting distortion functions and their associated parameters. However, we first introduce a measure of the magnitude of probability density displacement that results from the application of a distortion function. This measure of density translation uses the median of the risk distribution, namely that point at which the undistorted distribution is partitioned with equal density on either side. After distorting the original risk distribution, the magnitude of density translated from the left of the undistorted median to the right of the median using distortion function g is computed as

$$R_g \equiv \frac{(g \circ S)(\psi)}{S(\psi)}, \tag{12}$$

where $\psi \equiv \inf\{x > 0 : S(x) = 0.5\}$ denotes the median of the undistorted risk distribution, S is the undistorted survivor function, and $(g \circ S)$ is the distorted survivor function. Since the distortion functions used in this research all shift density to the right, we see that

$$1 \leq R_g \leq 2,$$

since by this ratio measurement all of the density to the left of the median can theoretically be shifted to the right of the median. However, R_g does not measure how "far" this density has been shifted – it only reflects the fact that it has been translated beyond the undistorted median. Conversely, if $R_g = 1$, this implies that no distortion has been applied whatsoever.

3.1 Effectiveness and Efficiency

The primary risk measure considered in this work is the expectation of the risk random variable. Recall that expectation has a drawback in that low-frequency risk values tend to be "dampened out" by the values with the greatest relative frequency. However, distortion functions can provide the decision maker with the ability to control expectation to predictable degrees. In choosing a distortion function for a specified risk distribution, the decision maker would like to know how *effective* each candidate distortion function/parameter combination is in inflating the expectation risk measure. After applying distortion g and computing the distorted expectation $\mu_g \equiv \hat{E}(X)$, the measures can be compared to determine which distortion has the greatest effect on that risk distribution's mean. To develop the idea further, we define the following measure.

Definition 1 *The effectiveness of a distortion function is defined as the ratio,*

$$K = \mu_g / \mu_0, \tag{13}$$

where μ_g is the distorted risk measure obtained by applying distortion g, and μ_0 is the undistorted risk measure.

This ratio can, for example, be used to directly compare a unique distortion function/parameter combination over different distributions, measuring that combinations's effectiveness in changing each distribution's expectation as a percentage increase. Because $\mu_g \geq \mu_0$ we see that $K \geq 1$, and whenever $K = 1$, the risk distribution is undistorted. Similarly, two different distortion function/parameter combinations applied to two dissimilar risk distributions having equal K-values are deemed to be equally effective in distorting (increasing) the expectation risk measure.

Through numerical experimentation, we have observed clear contrasts in the way different distortion function/parameter pairs shift density. As applied to a single risk distribution, one combination may require significant density shift before its K-value matches that of another pairing which has a greater effect on the distribution's tail. Prototypical examples are the PH and DP distortions. The PH distortion accumulates density in the right tail while the DP accumulates it closer to the mode, so the PH generally has a greater effect on expectation. A measure to reflect the magnitude of density shift has already been established, namely R_g. For this reason, it seems beneficial to combine the two measures K and R_g into a single measure of *efficiency.*

Definition 2 *The efficiency of a distortion function g is defined as the ratio of the normalized change in the risk measure to the normalized change in density given by*

$$E = K/R_g. \tag{14}$$

The measure E should not be confused with the concept of statistical efficiency related to parameter estimation. Intuitively, if a distortion function/parameter combination has a large effect on the expectation risk measure while shifting a relatively small magnitude of density, then that pairing is highly efficient when applied to the given distribution.

One might ask, "Why would a decision maker care about the magnitude of density being shifted? Why isn't the effectiveness of the distortion function/parameter combination all he or she needs to know in making a selection?" Note that without the efficiency measure, there would be no need to distinguish between two pairings with identical effectiveness; the decision maker might conclude that one is just as good as the other, even though the underlying distribution is being changed in an entirely different manner depending on the choice. As an example, consider Figure 2 which shows an undistorted Weib(2,2) distribution along with its PH ($a = 0.2$) and DP ($b = 31$) distortions. Both of the distorted distributions have $K \approx 2.24$, but the densities are hardly similar.

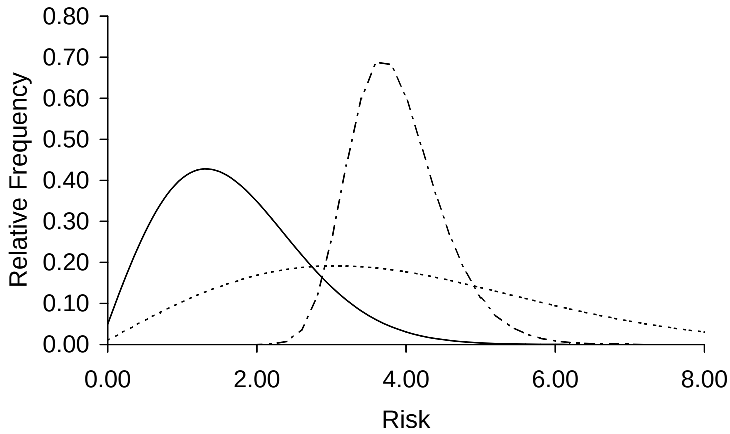

Figure 2: Distorted densities when X ~Weib(2,2) with distortion parameters $a = 0.2$ and $b = 31$ (solid is no distortion, $\cdots\cdot$ PH, $- \cdot -\cdot$ DP).

Furthermore, the decision maker should care a great deal about the magnitude of shifted density required to achieve a desired increase in the resulting risk measure. More specifically, the decision maker has approximated risk distributions using the inputs of subject matters experts who presumably possess expertise the decision maker is lacking. For every unit increase of the measure R_g, the decision maker is taking an additional “step” away from the recommendations of his or her advisors (and the assumed true distributions). To illustrate this point, consider again Figure 2 in which the distortions have transformed the original Weibull risk distribution into two radically different ones. Thus, it seems likely that the decision maker would prefer one of two possible courses of action in choosing a distortion function/parameter combination:

1. Achieve the maximum increase in the expectation while affecting the original risk

distribution by (no more than) a specified amount; or

2. achieve a specified increase in expectation while altering the original risk distribution as little as possible.

In either case, efficiency is the measure which provides the appropriate answer.

In order to investigate the impact of the distortion parameters on these measures, we carried out a 3^k-factorial designed experiment. For this purpose, we arbitrarily selected the following distribution parameters for each of the four distributions noted earlier: Exp(3.5), Weib(2,2), Tria(1,7,4), and $U(1,7)$. The factorial design was used to study the effects of each parameter (a, b, and c) in the gamma-beta distortion, and within this factorial design, each of the involved parameters was required to have relatively equal power over the R_g measure so that the interaction effects could be analyzed in a "fair" manner. Since we chose a face-centered cube design, three equally-spaced values were used for each parameter. Table 5 summarizes the selected distortion parameter values (or treatments). Recall that when $a = 1$, $b = 1$, and $c \to \infty$, no distortion is applied. We note that distortion is inversely proportional to the parameters a and c while it is proportional to the parameter b.

Table 5: Selected distortion parameter treatments.

Distortion (Parameter)	Selected Values		R_g (% density shift)
Proportional Hazard (a)	High	0.90	1.07 (7%)
	Mid	0.75	1.19 (19%)
	Low	0.60	1.32 (32%)
Dual Power (b)	Low	1.10	1.07 (7%)
	Mid	1.30	1.19 (19%)
	High	1.50	1.29 (29%)
Exponential (c)	High	3.60	1.07 (7%)
	Mid	2.20	1.11 (11%)
	Low	0.80	1.30 (30%)

Table 6 records the efficiency and effectiveness measures for the risk distributions and single-parameter distortions studied in this paper. In general, as the amount of distortion is increased, the efficiency is decreased. There are three exceptions to the general rule, however, and the efficiency measures for these three cases are highlighted in bold face type. Specifically, in the case of the exponential distribution, efficiency increases with distortion when using the PH and EX distortion functions. For the PH distortion applied to the Weibull distribution, efficiency at first decreases as distortion is increased, then begins to increase again. A brief investigation to verify this result showed that the least efficiency occurs at about $a = 0.72$.

Table 6: Effectiveness and efficiency measures for all distortion/distribution pairings.

Distortion →	PH			DP			EX		
Measure ↓	$a = 0.9$	$a = 0.75$	$a = 0.6$	$b = 1.1$	$b = 1.3$	$b = 1.5$	$c = 3.6$	$c = 2.2$	$c = 0.8$
Exp(3.5), $\mu_0 = 0.285714$									
μ_g	0.3175	0.3810	0.4762	0.3036	0.3363	0.3658	0.3058	0.3189	0.3791
R_g	1.0718	1.1892	1.3193	1.0670	1.1877	1.2929	1.0693	1.1131	1.3027
K	1.1111	1.3333	1.6667	1.0625	1.1772	1.2803	1.0704	1.1161	1.3270
K/R_g	**1.0367**	**1.1212**	**1.2633**	0.9958	0.9911	0.9903	**1.0010**	**1.0027**	**1.0186**
Weib(2,2), $\mu_0 = 1.772454$									
μ_g	1.8683	2.0467	2.2882	1.8448	1.9713	2.0788	1.8449	1.8911	2.0971
R_g	1.0719	1.1895	1.3199	1.0670	1.1879	1.2932	1.0694	1.1133	1.3032
K	1.0541	1.1547	1.2910	1.0408	1.1122	1.1729	1.0408	1.0669	1.1831
K/R_g	**0.9834**	**0.9707**	**0.9781**	0.9755	0.9362	0.9069	0.9733	0.9584	0.9079
Tria(1,7,4), $\mu_0 = 4.000$									
μ_g	4.1163	4.3218	4.5777	4.1033	4.2793	4.4246	4.0971	4.1586	4.4275
R_g	1.0718	1.1892	1.3193	1.0670	1.1877	1.2929	1.0693	1.1131	1.3027
K	1.0291	1.0804	1.1444	1.0258	1.0698	1.1062	1.0243	1.0396	1.1069
K/R_g	0.9602	0.9086	0.8675	0.9614	0.9007	0.8556	0.9579	0.9340	0.8497
$U(1,7)$, $\mu_0 = 4.000$									
μ_g	4.1579	4.4285	4.7500	4.1428	4.3913	4.6000	4.1387	4.2265	4.6093
R_g	1.0718	1.1892	1.3193	1.0670	1.1877	1.2929	1.0693	1.1131	1.3027
K	1.0395	1.1071	1.1875	1.0357	1.0978	1.1500	1.0347	1.0566	1.1523
K/R_g	0.9699	0.9310	0.9001	0.9707	0.9243	0.8895	0.9676	0.9492	0.8846

Using Table 6, some general rules (within the limits of this study) can be established for selecting a distortion function to apply to a distribution. Recall that a decision maker would likely be interested in either (i) achieving the largest possible increase in the mean given a specified maximum shift in density, or (ii) shifting the density by the smallest amount required to achieve a specified increase in expectation. Using Table 6, some answers may be available when objective (i) is of primary importance. Table 7 was created from Table 6 by comparing efficiency across categorized values of R_g. For example, considering the triangular distribution in Table 6, the low-distortion efficiency values are 0.9602 for the PH

($a = 0.9$), 0.9614 for the DP ($b = 1.1$), and 0.9579 for the EX ($c = 3.6$). Since the DP value is the highest, this was entered into the appropriate cell of Table 7. Thus in the case of objective (i) when assuming a triangular risk distribution, the DP distortion is the most efficient (although the values are relatively close in this case).

In examining Table 7, note once again that the difference in the R_g values between the PH and DP distortions ($R_g \approx 1.19$) and the EX distortion ($R_g \approx 1.11$) at the "moderate" distortion level could be significant in the final selection of a distortion function at that level. In addition, note that decision maker objective (ii) could be answered just as easily as objective (i), but the original response surface study which facilitated the distortion parameter choices would have had to fix the distorted *expectations* rather than the *amount of density shift* being applied.

Table 7: Suggested distortions for selected distributions (via efficiency).

Risk Distribution	Low Distortion (0-10%)	Moderate Distortion (11-20%)	Heavy Distortion (21-30%)
Exp(3.5)	PH	PH	PH
Weibull(2,2)	PH	PH	PH
Tria(1,7,4)	DP	EX	PH
$U(1,7)$	DP	EX	PH

3.2 Some Distortion Selection Guidelines

In this subsection, we summarize some conclusions that can be drawn from our simple designed experiment regarding the selection of distortion functions and their associated parameters. It is important to note that these results are not generalizable to all risk scenarios, but illustrate the means by which one might make such selections for the distributions we have considered here. More specifically, our conclusions are valid especially when the highly adverse outcomes correspond to the right tail of the distribution (i.e., when the decision maker desires to shift probability density to the right).

1. When an exponential or Weibull distribution is appropriate for the risk scenario, the PH distortion is drastically more efficient than the DP or EX. In the case of the exponential distribution, the PH also leaves the mode in place at zero, while other distortions "pull" the mode away from zero.

2. For the triangular and uniform distributions, no distortion appears to be as totally dominant (in efficiency) as the PH is for the exponential and Weibull. For each of these bounded distributions, the DP distortion is the most efficient in cases where only a small amount of distortion is required; questionably, the EX is more efficient in the vicinity of $R_g = 1.15$; and the PH is most efficient when larger amounts of distortion are required.

3. If higher moments are desired from the distorted distribution (e.g., the variance may well be of concern), then the DP and EX distortions may be preferred over the PH. Particularly in the case of the Weibull and triangular distributions, the DP accumulates density near the mean, likely reducing the impact on variance.

4. The parameter b of the DP distortion has a meaningful interpretation. In particular, it corresponds to the expected value of the worst outcome when b samples are taken from the random variable [14]. If the decision maker appreciates this interpretability but wishes to use either the PH or EX distortion, a value of b can be obtained which results in a DP match in μ_g to the specified a or c parameter. In this manner, the interpretability can be "loaned" to the PH and EX distortions through a single extra step.

In section 4, we illustrate the means by which distorted risk measures may be employed to incorporate the risk of capability shortfalls in a resource allocation problem.

4 Resource Allocation and Distortion: An Illustration

Suppose there are nine distinct areas of military (specifically fighter aircraft) capability that might be of interest to a decision maker. Table 8 provides the descriptions of such notional areas.

Table 8: Notional aircraft capability areas and descriptions.

Area (i)	Name	Description
1	Reconnaissance	Locate specific areas and record information about those areas
2	Range and Payload	Combat radius, weapon types and quantities
3	Communications	Ability to transmit and receive messages
4	Passive Sensors	Detection ability without broadcasting a radio frequency (RF) signal
5	Offensive Firepower	Ability to employ armament against ground and airborne targets
6	Self-Defense	Ability to defend against infrared- and RF-guided threats
7	Life Support	Ability to protect the pilot and provide for human needs (e.g., oxygen)
8	Networking	Ability of the aircraft to integrate into the battle space
9	Availability	The proportion of time the aircraft is available for its missions

A shortfall in capability area i creates a risk (say X_i) with undistorted risk measure $E(X_i)$ and distorted risk measure $\hat{E}(X_i)$, $i = 1, 2, \ldots, 9$. However, to address shortfalls in capability (i.e., to mitigate risk), six distinct risk-mitigating systems may be acquired as summarized in Table 9.

Table 9: Potential risk-mitigating systems.

System (j)	Description
1	Color cockpit display
2	Enhanced mission computer
3	New air-to-ground weapons system
4	New helmet-mounted targeting system
5	Enhanced digital radar system
6	Improved ground support equipment

Let $m_{i,j}$ denote the shortfall mitigation to area i ($i = 1, 2, \ldots, 9$) obtained from acquisition of system j ($j = 1, 2, \ldots, 6$). For instance, if $m_{i,j} = 0.50$, then the purchase of system j reduces the shortfall in capability area i by 50%. We initially assume that the decision maker can choose to acquire some or all of a risk-mitigating system. Let x_j represent the proportion of system j to be acquired so that $0 \leq x_j \leq 1$, $j = 1, 2, \ldots, 6$ (i.e., partial investments are permissible without loss of contribution). Next, define c_j as the cost of acquiring one complete unit of system j. The decision maker's objective is to maximize the risk mitigation by strategically choosing the proportion of various systems to purchase, subject to a fixed budget B. The optimal strategy may be determined by solving the linear program (LP),

$$\max \quad \sum_{i=1}^{9} \sum_{j=1}^{6} \hat{E}(X_i) m_{i,j} x_j \tag{15a}$$

$$\text{s.t.} \quad \sum_{j=1}^{6} c_j x_j \leq B \tag{15b}$$

$$0 \leq x_j \leq 1, \quad j = 1, 2, \ldots, 6. \tag{15c}$$

An intuitive explanation of the LP is as follows. Our decision maker would like to selectively apply distortion to the various risk distributions in order to better reflect his or her own risk priorities. For this reason, in (15a), we weight each $m_{i,j}$ by the distorted expectation risk measure, $\hat{E}(X_i)$, $i = 1, 2, \ldots, 9$. If we instead use the undistorted values, $E(X_i)$, $i = 1, 2, \ldots, 9$, this corresponds to solving the problem using only the information provided by subject matter experts based solely on the unadjusted expected risk (i.e., ignoring the decision maker's concerns about catastrophic loss). If we set $\hat{E}(X_i) = 1$ for each i in (15a), this is equivalent to ignoring the risks altogether.

With regard to the budgetary constraint (15b), we have assumed in this example that the cost of acquiring system j is linear in the proportion of investment in system j. Of course, it is possible that the costs may differ if funding of only part of a risk-mitigating system is selected. A number of approaches in the literature are available to model various operational settings if other conditions apply (see [6] among others). Nonetheless, we solve the LP under this assumption for the purpose of illustrating the methodology. It is worth noting that Woodward [15], used an integer programming approach and a distorted expectation risk measure, but uniformly applied the DP distortion function with a constant parameter b across all capability areas. Our approach permits the decision maker to vary the type and level of distortion in each area while also allowing for the possibility of partial investments when appropriate.

We now illustrate solving the LP in a specific problem instance. We assume the budget is fixed at $B = 25$ monetary units. Suppose the decision maker is least risk averse to catastrophic loss in capability areas 8 and 9, somewhat risk averse to catastrophic loss in areas 4, 6, and 7, and most risk averse to catastrophic loss in areas 1, 2, 3, and 5. The weights summarized in Table 10 reflect the decision maker's degree of risk aversion in each area. In particular, a higher weight represents greater risk aversion. The nine distributions, assumed to originate from the inputs of nine teams of subject matter experts, are also included. Specifically, these distributions correspond to the risk associated with shortfalls in the respective capability areas summarized in Table 8.

Table 10: Notional data for illustrative example.

Area (i)	Weight	Distribution	$\mu_0 = E(X_i)$
1	20	Weib(3.5,3.3)	2.9692
2	30	Tria(0,4.67,3.2)	2.6233
3	19	$U(0,4)$	2.0000
4	13	Tria(0,4,2)	2.0000
5	46	Weib(2.04,1.74)	1.5416
6	6	Weib(3.08,2.84)	2.5391
7	6	$U(1,3)$	2.0000
8	0	Exp(0.45)	2.2222
9	0	Tria(0,1.875,0.5)	0.7917

We assume the decision maker would like to impose one of two priorities: (i) obtain the greatest possible increase in the expectation risk measure given a specified shift in density, or (ii) minimize the magnitude of density shifted to achieve a specified increase in the expectation. While either priority may be considered, we will proceed with this example on the assumption that the decision maker prefers objective (i), and that the degree of risk aversion (i.e, the weights in Table 10) assigned to an area corresponds to a specific shift in density R_g. For instance, for Area 1 (reconnaissance capability), the risk distribution is assumed to be Weib(3.5,3.3), and the decision maker has chosen, based on available information and projections, to shift 20% of the density beyond the median, or $R_g \approx 1.20$. At this level, using this distribution, the PH distortion is the most efficient. Setting $a = 0.735$ results in a distorted expectation of $\mu_g = 3.2422$. We continue in this fashion for all risk distributions, applying distortions based on the recommendations of Table 7. Table 11 summarizes the results of selectively distorting as per the pre-specified preferences of the decision maker. The column entitled, "Distortion" is the selected distortion function and its associated parameter value.

Table 11: Selection of distortion functions and parameters.

Area (i)	Distribution	R_g	Distortion	$\mu_g = \hat{E}(X_i)$
1	Weib(3.5,3.3)	1.20	PH, $a = 0.735$	3.2422
2	Tria(0,4.67,3.2)	1.30	PH, $a = 0.062$	3.0197
3	$U(0, 4)$	1.19	EX, $c = 1.300$	2.2539
4	Tria(0,4,2)	1.13	EX, $c = 1.900$	2.1223
5	Weib(2.04,1.74)	1.46	PH, $a = 0.450$	2.2801
6	Weib(3.08,2.84)	1.06	PH, $a = 0.915$	2.6134
7	$U(1, 3)$	1.06	DP, $b = 1.090$	2.0431
8	Exp(0.45)	1.00	N/A	2.2222
9	Tria(0,1.875,0.5)	1.00	N/A	0.7917

Table 12 summarizes the impact of each system on mitigating capability shortfalls. The table elements correspond to the percent shortfall mitigation that each potential acquisition addresses in all nine areas. For example, system 2 mitigates the risk of a shortfall in area 1 by 19%.

Table 12: Percent shortfall mitigation.

Area (i)	$m_{i,1}$	$m_{i,2}$	$m_{i,3}$	$m_{i,4}$	$m_{i,5}$	$m_{i,6}$
1	0.00	0.19	0.00	0.26	0.26	0.00
2	0.46	0.21	0.00	0.12	0.00	0.00
3	0.34	0.19	0.00	0.05	0.00	0.23
4	0.14	0.42	0.00	0.36	0.00	0.05
5	0.10	0.21	0.92	0.30	0.00	0.10
6	0.00	0.00	0.54	0.00	0.00	0.11
7	0.16	0.00	0.00	0.00	0.25	0.00
8	0.19	0.00	0.00	0.00	0.31	0.48
9	0.00	0.00	0.00	0.00	0.33	0.36
c_j	7.0	7.0	10.0	8.0	8.0	9.0

The last row of Table 12 is the cost associated with the purchase of a *complete* system.

Assuming a budget of 25 units, Table 13 summarizes the optimal solution to the resource allocation problem when formulated as an LP. The "Unweighted" solution assumes that we do not weight the mitigation terms by a risk measure at all. The rows entitled, "Weighted, Undistorted" and "Weighted, Distorted" provide the solutions using undistorted and distorted risk measures, respectively. In Table 13, an entry of 1.0 represents a recommendation to purchase a complete system, a decimal represents a partial purchase, and 0.0 represents no purchase.

Table 13: Optimal solutions under various scenarios (LP formulation).

Weighting Scheme	x_1	x_2	x_3	x_4	x_5	x_6
Unweighted	1.000	1.000	0.200	0.000	0.000	1.000
Weighted, Undistorted	1.000	1.000	0.300	1.000	0.000	0.000
Weighted, Distorted	1.000	1.000	1.000	0.125	0.000	0.000

The different optimal solutions for the three scenarios agree with intuition. In particular, we note that systems 1 and 2 are consistently chosen because they significantly mitigate the risk of shortfalls in capability areas 2-5 which the decision maker has deemed to be relatively important areas. On the contrary, systems 5 and 6 are seldom chosen due to the fact that they do not impact the critical areas (2-5) in a significant way. System 3 becomes an important asset once the decision maker's preferences are included because it has the potential to mitigate risks in a shortfall of capability area 5, the area in which the decision maker is most risk averse.

For the sake of comparison, we next consider a binary integer programming (BIP) formulation of the problem where constraint (15c) is replaced by the constraint $x_j \in \{0, 1\}$, $j = 1, 2, \ldots, 6$. Table 14 summarizes the optimal solutions in this case.

Table 14: Optimal solutions under various scenarios (BIP formulation).

Weighting Scheme	x_1	x_2	x_3	x_4	x_5	x_6
Unweighted	1.000	0.000	1.000	0.000	1.000	0.000
Weighted, Undistorted	1.000	0.000	1.000	1.000	0.000	0.000
Weighted, Distorted	1.000	0.000	1.000	1.000	0.000	0.000

As expected, when partial investments are prohibited, the optimal solutions differ (see for example [9]). However, we note that in this illustrative binary IP formulation, systems 5 and 6 remain unimportant in mitigating the risk of capability shortfalls while system 1 is *always* selected when the risk measures are included. When choosing between the LP, BIP or a mixed model, of course, the analyst will select the model that most accurately represents the decision environment. In section 5, we provide some concluding remarks and possible directions for future inquiry.

5 Conclusions

The properties of distortion functions have been well documented in the current risk analysis literature. However, the appropriate selection of a distortion function and its corresponding parameters is a problem that has not received much attention. This study takes an initial step toward addressing this important issue. Our primary objective was to provide some practical recommendations for risk analysts who seek to use distortion functions to adjust the expectation risk measure to better account for low-likelihood yet potentially catastrophic events. For our purposes, we considered risks that may arise from shortfalls in military or homeland security capabilities which may result in obvious detrimental outcomes.

We have provided a procedure, via analytical and empirical methods, for the selection of distortion functions and their parameters on a set of parametric risk distributions. The use of distortion functions provides a tractable and documentable procedure to investigate the shifting of risk in the face of catastrophic events. Two new measures, efficiency and effectiveness, were proposed to distinguish the effects of different distortions and to make basic recommendations regarding the appropriateness of certain distortion functions and parameters using specific risk distributions. Additionally, a linear programming model was formulated to illustrate the means by which the distorted expectation risk measure can be used to influence the acquisitions plan of a risk-averse decision maker.

There are some obvious shortcomings in this work which are noted here. First, the selection guidelines that we provide are limited to the risk distributions considered in this study. Of course, it will be important to consider a wider range of distributions and to study the interaction between distribution and distortion function parameters. Moreover, we considered only the expectation risk measure, and it may prove useful to consider other coherent measures in the future. Another limitation stems from the use of the quantity R_g which measures only the magnitude of density shifted beyond the median of the undistorted risk distribution. This measure really does not tell us how "far" beyond the median the density has been translated. Other measures should be considered, as should a more comprehensive risk measure such as those described in [8]. It may also be instructive to investigate the relationship between the skewness of the risk distribution (perhaps using Pearson's skewness coefficient) and either the percent change in expectation or R_g. There appears to exist some correlation between Pearson's coefficient and the normalized mean. That is, the mean of some risk distributions (the exponential is one case) is more sensitive to the application of distortion than others. Finally, further research regarding the effects of distortion on variance may significantly impact the selection of distortion functions for specific risk scenarios.

Acknowledgements

The authors thank two anonymous referees for their insightful comments and suggestions. We also wish to thank Dr. Edward Pohl who acted as Editor for this paper.

References

[1] Artzner, P., Delbaen, F., Eber, J.-M., and D. Heath (1997). Thinking coherently. *Risk*, 10: 68-71.

[2] Artzner, P., Delbaen, F., Eber, J.-M., and D. Heath (1999). Coherent measures of risk. *Mathematical Finance*, 9: 203-228.

[3] Benati, S. (2003). The optimal portfolio problem with coherent risk measure constraints. *European Journal of Operational Research*, 150: 572-584.

[4] Clemen, R. T. and T. Reilly (2001). *Making Hard Decisions with Decision Tools Suite*, 2nd Edition. Pacific Grove, CA: Duxbury.

[5] Lambert, J.H., Matalas, N.C., Ling, C., Haimes, Y.Y., and D. Li (1994). Selection of probability distributions in characterizing risk of extreme events. *Risk Analysis*, 14 (5): 731-742.

[6] Martello, S. and P. Toth (1990). *Knapsack Problems: Algorithms and Computer Implementations*, John Wiley & Sons, NY.

[7] McLeish, D. L., and R. M. Reesor (2003). Risk, entropy, and the transformation of distributions. *North American Actuarial Journal*, 7: 128-144.

[8] Sarin, R. K., and M. Weber (1993). Risk-value models. *European Journal of Operational Research*, 70: 135-149.

[9] Taha, H.A. (1975). *Integer Programming: Theory, Applications, and Computations*, Academic Press.

[10] Wang, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics*, 17: 43-54.

[11] Wang, S. (1996a). Premium calculation by transforming the layer premiun density. *ASTIN Bulletin*, 26: 71-92.

[12] Wang, S. (1996b). Ordering of risks under PH-transforms. *Insurance: Mathematics and Economics*, 18: 109-114.

[13] Wang, S. S., Young, V. R., and H. H. Panjer (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics*, 21: 173-183.

[14] Wirch, J. L., and M. R. Hardy (1999). A synthesis of risk measures for capital adequacy, *Insurance: Mathematics and Economics*, 5: 337-347.

[15] Woodward, W. E. (2004). *Measuring the Risk of Shortfalls in Air Force Capabilities.* M.S. Thesis, Department of Operational Sciences, Air Force Institute of Technology, Dayton, OH.