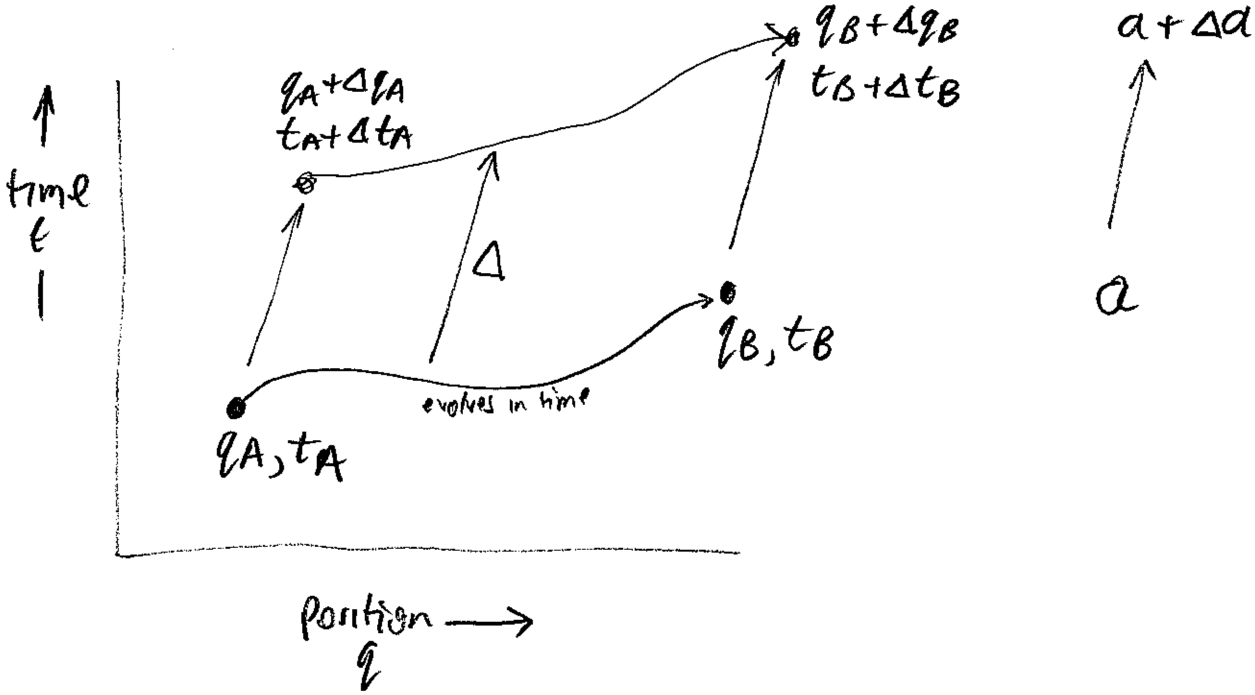
Ehrenfest's Proof that $2\bar{T}/\nu$ is an adiabatic invariant

Amsterdam Academy, 1917, pp. 576–97. Appendix

(simplified to 1 dimension q, a)

System with Lagrangian
$$L = \underbrace{T(q, \dot{q}, a)}_{\text{kinetic energy}} - \Phi(q, a)$$
$\dot{q}$: d/dt

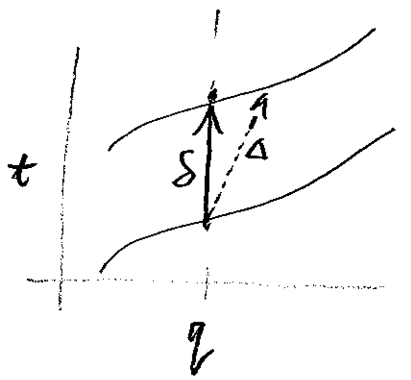
a = external mechanical variable to be changed slowly e.g. external field, position piston ...



compute change in action $\int_{t_A}^{t_B} L\,dt$ under Δ

$$\Delta\int_{t_A}^{t_B} L\,dt = L_B\Delta t_B - L_A\Delta t_A + \int_{t_A}^{t_B}\delta L\,dt$$

δ = changes for simultaneous motions



$$\delta L = \frac{\partial L}{\partial q}\cdot\delta q + \frac{\partial L}{\partial \dot q}\underbrace{\delta\dot q} + \frac{\partial L}{\partial a}\Delta a$$

$$\delta\dot q = \delta\left(\frac{dq}{dt}\right) = \frac{d}{dt}(\delta q)$$

$$\int_{t_A}^{t_B}\delta L\,dt = \underbrace{\int_{t_A}^{t_B}\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial\dot q}\frac{d}{dt}(\delta q)\,dt}_{\text{integrate by parts}} + \Delta a\int_{t_A}^{t_B}\frac{\partial L}{\partial a}\,dt$$

$$= \int_{t_A}^{t_B}\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q}\delta q\right)} + \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right)\delta q\,dt$$

$$\left.\frac{\partial L}{\partial\dot q}\right|_B \delta q_B - \left.\frac{\partial L}{\partial\dot q}\right|_A \delta q_A$$

$$p \overset{\text{def}}{=} \frac{\partial L}{\partial\dot q}$$

$$\Delta q = \delta q + \frac{dq}{dt}\Delta t$$

$$= \delta\dot q + \dot q\Delta t$$

combine

$$\Delta\int_{t_A}^{t_B} L\,dt = \Big(\Delta t\underbrace{\{L-p\dot q\}}_{-E}\Big)_A^B + \Big(p\,\Delta q\Big)_A^B + \int_{t_A}^{t_B}\underbrace{\left(\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)}_{\text{vanishes.}}\delta q\,dt + \int_{t_A}^{t_B}\frac{\partial L}{\partial a}dt\cdot\Delta a$$

constant energy on process for fix a

JDN conserved Hamiltonian $H = p\dot q - L$

vanishes. Newton's equation motion

$-\frac{\partial L}{\partial a} =$ "constant force that must act to keep a constant" $= -A$

mean force $= \frac{1}{t_B-t_A}\int\frac{\partial L}{\partial a}dt$

Hence

$$\Delta\int_{t_A}^{t_B}\underbrace{(T-\Phi)}_{L}dt = -E.\Delta(t_B-t_A) + (t_B-t_A)\bar A\,\Delta a + p_B\Delta q_B - p_A\Delta q_A \quad (k)$$

Assume T is quadratic in $\dot q$

$T = A\dot q^2$ Then $E = p\dot q - L$

$\frac{\partial T}{\partial \dot q} = 2A\dot q$

$$= \dot q\frac{\partial L}{\partial \dot q} - (T-\Phi)$$

$$= \underbrace{2A\dot q^2}_{2T} - T + \Phi = T + \Phi$$

$$\Delta\int_{t_A}^{t_B}(T+\Phi)dt \overset{?}{=} \Delta\Big[E(t_B-t_A)\Big] = (t_B-t_A)\Delta E + E\,\Delta(t_B-t_A) \quad \cdots \quad (l)$$

Sum (k) & (l)

$$\Delta\int_{t_A}^{t_B}2T\,dt = (t_B-t_A)(\Delta E + \bar A\,\Delta a) + \underbrace{p_B\Delta q_B - p_A\Delta q_A}_{\text{vanishes for a periodic motion}}$$

Hence for very slow change
in parameter $a \to a + \Delta a$

$$\Delta \int_{t_A}^{t_B} 2T\,dt = (t_B - t_A)\underbrace{(\Delta E + \bar{A}\,\Delta a)}_{\uparrow \text{ For v.small } \Delta a} = 0$$

$$\Delta E = -\bar{A}\,\Delta a$$

Hence $\underbrace{\int_{t_A}^{t_B} 2T\,dt}$ is an adiabatic invariant

2 mean kinetic energy $2\bar{T} = \underbrace{\frac{1}{t_B - t_A}}_{\text{frequency } \nu} \int_{t_A}^{t_B} 2T\,dt$

since $t_B - t_A$ = time for one cycle

$\therefore \dfrac{2\bar{T}}{\nu}$ is an adiabatic invariant

How does Ehrenfest recover

Force acting when a changes $= A = -\dfrac{\partial L}{\partial a}$??

$$\text{Force} = A = \frac{\partial}{\partial a} E = \frac{\partial}{\partial a}\left(p\dot{q} - L\right) = -\frac{\partial L}{\partial a}$$

<u>**Iff**</u> $0 = \dfrac{\partial}{\partial a}\left(p\dot{q}\right) = \dfrac{\partial}{\partial a}\left(\dfrac{\partial L}{\partial \dot{q}}\dot{q}\right) = \dfrac{\partial}{\partial a} T$

Obtains if $T = A(q)\,\dot{q}^2$

$\uparrow$ No a dependence