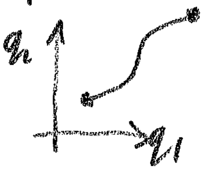
Hamilton-
Jacobi
Theorey

Hamilton–Jacobi Theory in a Nutshell

Lagrangian formulation

In configuration space



motion makes $\int_{t_1}^{t_2} L\,dt$ extremal

$L = L(q_i, \dot{q}_i, t)$

$$H = \sum_i p_i \dot{q}_i - L$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Hamiltonian formulation

In phase space, $p_i\, q_i$

motion satisfies

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

for Hamiltonian

$$H = H(q_i, p_i, t)$$

main result

"Hamilton-Jacobi Equation"

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0$$

"Hamilton's principle function S"

$$S = \int_{t_0}^{t} L(q_i, \dot{q}_i, t)\,dt + \text{constant}$$

Finding S is enough to solve problems which motions are allowed

Special case: H independent of t

$$H(q_i, p_i, t) = H(q_i, p_i)$$

$\therefore$ $\frac{\partial S}{\partial q_i}$ is independent of t

$$\therefore \quad S = W(q_i, \alpha_i) - Et$$

constants of motion

Energy since $\frac{\partial S}{\partial t} = H = E$

"Hamilton's characteristic function"

$$\underbrace{H\left(q_i, \frac{\partial W}{\partial q_i}\right)}_{\text{independent of } t} = E$$

source: Goldstein, Classical Mechanics

Lagrange Principle

System characterized by Lagrangian $L(q_i, \dot{q}_i, t)$

q_i: configuration space variables

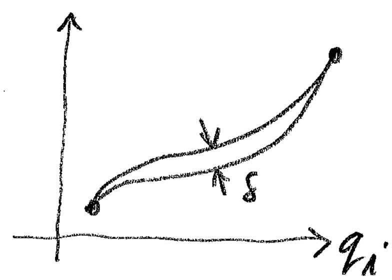
$\dot{} = \frac{d}{dt}$

motion extremizes

$$\int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$

t_1, t_2 fixed
endpoints $q(t_1), q(t_2)$ fixed

Variation operator δ
= difference between neighboring trajectories



$$0 = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial t}\cancelto{0}{\delta t}\; dt$$

$$\frac{\partial L}{\partial \dot{q}_i}\delta\left(\frac{dq_i}{dt}\right) = \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)$$

$$\therefore 0 = \underbrace{\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\Bigg|_{t_1}^{t_2}}_{\substack{0 \text{ since} \\ \delta q_i(t_1) = \delta q_i(t_2) = 0}} + \int_{t_1}^{t_2} \underbrace{\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)}_{\substack{\text{This must vanish since} \\ \delta q_i \text{ is arbitrary.}}}\delta q_i\, dt = 0$$

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0}$$

Euler-Lagrange equation

Hamilton's Principle

system is characterized by Hamiltonian $H(q_i, p_i, t)$ ⇐ Note we are now in a phase space with coordinates p_i, q_i

$p_i = \frac{\partial L}{\partial \dot{q}_i}$ ... but this fact not used here!

motion extremizes $p_i\dot{q}_i - H(q_i, p_i, t)$ in p-q phase space where start & end times, start & end p,q } fixed

$$0 = \delta \int_{t_1}^{t_2} \underbrace{p_i\dot{q}_i - H(p_i, q_i, t)}_{\text{"}f\text{"}}\, dt$$

where δ realizes variation

⇓ Euler-Lagrange equations in p-q space

$$\underbrace{\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right)}_{\frac{d}{dt}(p_i)} - \frac{\partial f}{\partial q_i} = 0 \qquad \underbrace{\frac{d}{dt}\frac{\partial f}{\partial \dot{p}_i}}_{0} - \frac{\partial f}{\partial p_i} = 0$$

⇓ ⇓

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}}$$

Hamilton's equations

Canonical transformations preserve Hamilton's Equations

p_i, q_i such that

$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$

[merely redescribe

$Q_i = Q_i(p_i, q_i, t)$

$P_i = P_i(p_i, q_i, t)$

$\Rightarrow$ Q_i, P_i such that

$\dot{P}_i = -\frac{\partial K}{\partial Q_i} \quad \dot{Q}_i = \frac{\partial K}{\partial P_i}$

for some new $K = K(P_i, Q_i, t)$

$\Downarrow$ Hence satisfy

$\delta \int_{t_1}^{t_2} p_i \dot{q}_i - H \, dt = 0$

$\Downarrow$ Hence satisfy

$\delta \int_{t_1}^{t_2} P_i \dot{Q}_i - K \, dt$

If we know this holds, we can get this if we define a new K by P_i, Q_i

$$\boxed{p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt}}$$

since

(i) the variation extremization is purely geometric & does not alter if we redescribe the trajectory using P, Q instead of p, q

(ii) F makes no contribution to the variation since $\int_{t_1}^{t_2} \frac{dF}{dt} dt = \underbrace{F(t_2) - F(t_1)}_{\text{same for all trajectories}}$

Choose different F $\Longrightarrow$ Generate different canonical transformations

e.g. case "2": $F = \underbrace{F_2(q_i, P_i, t)}_{}- Q_i P_i$

NB Note which variables appear here !!

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt}$$

becomes

$$p_i \dot{q}_i - H = -Q_i \dot{P}_i - K + \underbrace{\frac{d}{dt} F_2(q_i, P_i, t)}_{\frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_i} \cdot \dot{q}_i + \frac{\partial F_2}{\partial P_i} \cdot \dot{P}_i}$$

Rearrange:

$$\underbrace{\left(p_i - \frac{\partial F_2}{\partial q_i}\right)}_{}\dot{q}_i - H = \underbrace{\left(\frac{\partial F_2}{\partial t} - Q_i\right)}_{\text{vanishes}}\dot{P}_i - K + \frac{\partial F_2}{\partial t}$$

Equality must hold for all trajectories.

$\therefore$ This vanishes

$$\boxed{p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \qquad K = H + \frac{\partial F_2}{\partial t}}$$

Hence: Fix $F_2(q_i, P_i, t)$ $\Longrightarrow$ Fix a canonical transformation

Hamilton-Jacobi Equation

Choose a "case 2" transformation such that new Hamiltonian $K \equiv 0$

Then $\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0$ $\quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$ $\quad$ i.e. P_i, Q_i are constants of motion

$$K = H + \frac{\partial F_2}{\partial t}$$

becomes

$$H + \frac{\partial F_2}{\partial t} = 0$$

i.e.

$$\boxed{H\left(q_i, \frac{\partial F_2}{\partial q_i}, t\right) + \frac{\partial F_2}{\partial t} = 0}$$

since $p_i = \frac{\partial F_2}{\partial q_i}$ is one of the transformation equations

What use is it? Solving the H-J equation also solves the dynamical problem of the system's motion.

We find the constant P_i, Q_i that describe the motion.

Hamilton's characteristic Function W

specialise Hamilton-Jacobi Equation to case in which

(1) H is <u>not</u> dependent on time explicitly

(2) $F_2(q_i, P_i, t)$

P_i ↑ constants of motion

$P_i = \alpha_i$ (e.g. α_1 = energy)

write

$$F_2(q_i, P_i, t) = \underset{\uparrow}{S}(q_i, \alpha_i, t)$$

"Hamilton's Principal Function"

Hence H-J equation is solved by

$$F_2 = S(q_i, \alpha_i, t) = W(q_i, \alpha_i) - \alpha_1 t \Leftarrow$$

$\boxed{\text{Schrödinger Part II Equ. 2}}$

H-J equation : $H\left(q_i, \frac{\partial S}{\partial q_i}\right) + \underbrace{\frac{\partial S}{\partial t}}_{-\alpha_1} = 0$

since H is assumed independent of t, we must have $\frac{\partial S}{\partial q_i}$ is independent of t.

$\therefore S = \underset{\uparrow}{W} + \alpha t$

indep of t

$$\Downarrow$$

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1 \Leftarrow$$

energy

$\boxed{\text{Schrödinger Part II Equ. 1}}$

What is "Hamilton's Principal Function" S ?

① $S = F_2$ function used to generate canonical transformation to new constant Hamiltonian $K \equiv 0$

$$K = H + \frac{\partial F_2}{\partial t}$$

$$\therefore \quad \frac{\partial S}{\partial t} = K - H = -H$$

$p_i = \frac{\partial S}{\partial q_i}$ from transformation

② We work with the special case in which

$$S = S(q_i, \underbrace{\alpha_i}, t)$$

constants of motion can be set to the new P_i of the transformation

constants, since $\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$

combine

$$\frac{dS}{dt} = \sum_i \underbrace{\frac{\partial S}{\partial q_i}}_{p_i} \dot{q}_i + \sum_i \frac{\partial S}{\partial \alpha_i} \cancel{\dot{\alpha}_i}^{\,0} + \underbrace{\frac{\partial S}{\partial t}}_{-H} = \sum_i p_i \dot{q}_i - H = L$$

$$\boxed{\therefore \; S = \int L\, dt + \text{constant}}$$