# Kruskal-Szekeres Coordinates for a Schwarzschild Spacetime

Source: Robert Wald, *General Relativity*. Chicago: University of Chicago Press, 1984. p. 153.

Einstein and Rosen have identified the Schwarzschild radius as singular. To see that the singularities arise only as an artefact of the choice of coordinates used by them, we transform the coordinate representation of the metrical structure of the Schwarzschild spacetime to new Kruskal-Szekeres coordinates in which the singular behavior does not appear.

Einstein and Rosen use the spacetime coordinates $t$, $r$, $\theta$, $\phi$ in the expression for the Schwarzschild line element

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{1 - \frac{2m}{r}} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1}$$

The line element is simplified by using units for space, time and mass such that $c$=$G$=1. The Kruskal-Szekeres coordinates are defined implicitly by the relations

$$\left(\frac{r}{2m} - 1\right) e^{r/2m} = X^2 - T^2 \tag{2}$$

$$\frac{t}{2m} = ln\left(\frac{X+T}{X-T}\right) \tag{3}$$

The angle coordinates $\theta$ and $\phi$ are unchanged. They will have no special role in the analysis. The line element becomes

$$ds^2 = \frac{32m^3 e^{-r/2m}}{r}(dT^2 - dX^2) - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{4}$$

The Schwarzschild radius or event horizon is at $r = 2m$. It follows from (2) that the Schwarzschild radius at $r = 2m$ is given by

$$\left(\frac{r}{2m} - 1\right) = 0$$

From (2), the Schwarzschild radius corresponds to

$$X^2 - T^2 = 0 \qquad X = T \quad \text{and} \quad X = -T$$

These two straight lines in a coordinate plot define a wedge that enclosed the $X$ axis and has an apex at $X = T = 0$. The exterior of the black hole lies within this wedge, for the exterior is covered by the coordinates $r > 2m$ and $-\infty < t < \infty$. That is,

$$\left(\frac{r}{2m} - 1\right) > 0$$

It follows from (2) that this region corresponds to the wedge in Kruskal-Szekeres coordinates coordinates given by

$$X^2 - T^2 > 0 \qquad |X| > |T|$$

It follows from (2) that curves of constant $r > 2m$ are represented by hyperbolas

$$X^2 - T^2 = \text{constant} > 0$$

For curves of constant $r$, where $r$ is close to $2m$, so that

$$\varepsilon = \left(\frac{r}{2m} - 1\right)$$

is small, the hyperbolas are approximated as

$$X^2 - T^2 = \left(\frac{r}{2m} - 1\right) e^{\frac{r}{2m}} = \varepsilon e^{1+\varepsilon} \approx \varepsilon e$$

Curves of constant $t$ are given by (3)

$$ln\left(\frac{X+T}{X-T}\right) = \frac{t}{2m} = constant$$

It follows that

$$\left(\frac{X+T}{X-T}\right) = \left(\frac{\frac{X}{T}+1}{\frac{X}{T}-1}\right) = constant$$

from which we have that curves of constant $t$ are straight lines in the new coordinate system

$$T = \text{constant } X$$

Finally we have from (4) that lightlike trajectories given by $ds^2 = 0$ satisfy $(dT^2 - dX^2) = 0$.

Integrating we recover that lightlike trajectories satisfy

$$T = X + \text{constant} \quad \text{and} \quad T = -X + \text{constant}$$