

WHAT IS A NEWTONIAN SYSTEM? THE FAILURE OF ENERGY
CONSERVATION AND DETERMINISM IN SUPERTASKS

ABSTRACT. Supertasks recently discussed in the literature purport to display a failure of energy conservation and determinism in Newtonian mechanics. We debate whether these supertasks are admissible as Newtonian systems, with Earman and Norton defending the affirmative and Alper and Bridger the negative.

1. INTRODUCTION

Alper and Bridger (1988) have argued that certain “Newtonian” supertasks recently discussed in the literature (such as Perez Laraudogoitia, 1996; Earman and Norton, 1998) appear to violate energy conservation and to display indeterminism only because the analyses fail to account properly for the mathematical requirement of continuity of various dynamical quantities with time. Alper and Bridger’s conclusion depends upon a decision as to what counts as a Newtonian system. Definitions of continuity of differing stringency will determine whether the supertask systems in question are admissible as Newtonian systems. In Section 3 of this note, Earman and Norton will defend a more lenient definition that does admit the supertasks. In Section 4, Alper and Bridger will defend a more restrictive definition that does not admit the supertasks. The remaining sections are the results of discussions among the authors.¹

2. THE SUPERTASK ST

In Perez Laraudogoitia’s (1996) supertask, infinitely many, unit point masses P_1, P_2, \dots are arrayed at rest along the x -axis of space at positions $x = 1/2, 3/4, 7/8 \dots$. At time $t = 0$ a unit point mass P_0 , moving at unit velocity, passes $x = 0$ and approaches the remaining masses. Infinitely many perfectly elastic collisions ensue with the unit velocity passing in succession from P_0 to P_1 to P_2 etc. However, by $t = 1$, all collisions are completed and all masses are at rest. See Figure 1.



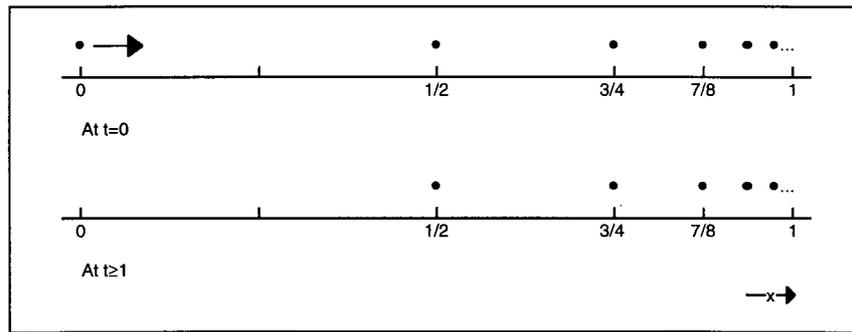


Figure 1. Supertask ST.

In the simplest analysis, each collision is instantaneous, so the velocity changes are instantaneous. This already threatens the Newtonian credentials of the supertask since, at the instant of collision, the velocity of the particles involved change discontinuously; that is, the velocity is not defined at that instant. The most natural solution is to assume that each collision is mediated by very strong short range forces that eliminate the discontinuities in the time dependence of velocities and provide them sufficient differentiability for the acceleration to be always defined.²

This supertask represents a violation of energy and momentum conservation. At $t = 0$ the total energy is $1/2$ in the form of kinetic energy and the total momentum is 1 . At $t = 1$, both quantities have dropped to zero. If we allow that the mechanics governing the supertask is time reversible, then the system displays indeterminism. The time reversed process is a system of particles at rest, which is spontaneously excited and emits a particle moving with velocity $v = -1$.

3. CHOOSING A MATHEMATICAL MODEL FOR THE SUPERTASK ST

A necessary condition for a system of particles to be Newtonian is that Newton's laws of motion are satisfied at all times. As applied to ST, if we define F_i , $i = 0, 1, 2, \dots$, as the force on particle P_i which is at position X_i and moves at velocity V_i , Newton's laws can be represented as the infinite set of coupled, linear first order equations:

$$(2.1) \quad F_i = \frac{dV_i}{dt} \quad V_i = \frac{dX_i}{dt}$$

with Newton's third law implemented as

$$(2.2) \quad \sum_{i=0}^{\infty} F_i = 0.$$

Alper and Bridger (1998) propose that these quantities be combined into three vectors

$$\mathbf{F} = (F_0, F_1, \dots), \quad \mathbf{X} = (X_0, X_1, \dots), \quad \mathbf{V} = (V_0, V_1, \dots)$$

each in its own infinite dimensional vector space. Equation (2.1) is now expressible in vectorial form as

$$(2.1') \quad \mathbf{F} = \frac{d\mathbf{V}}{dt}, \quad \mathbf{V} = \frac{d\mathbf{X}}{dt}.$$

By using the naturally defined scalar product, we can introduce a norm

$$\|\mathbf{V}\| = \sqrt{\sum_{i=0}^{\infty} V_i^2}$$

in the velocity vector space, so that space is converted to a (real) Hilbert space. If Newton's equations are to hold at all times t , then the velocity vector \mathbf{V} must be continuous at all t , including the supertask ending at time $t = 1$. If it fails to be continuous, it will also fail to be differentiable, and quantities in Equations (2.1) and (2.1') will not be defined at $t = 1$. A graph of the components $V_i(t)$ of $\mathbf{V}(t)$ suggests that continuity may fail at $t = 1$.

The top portion of Figure 2 shows the components plotted individually; they are plotted together in the bottom portion. Setting aside the effects of smoothing, each component has a value of either 0 or 1. In the interval $t = 0$ to $t = 1/2$, V_0 only is non-zero; in $t = 1/2$ to $t = 3/4$, V_1 only is non-zero; in $t = 3/4$ to $t = 7/8$, V_2 only is non-zero; etc. Thus, no matter how close we come to $t = 1$, there will always be a component whose value is 1.

The vector function $\mathbf{V}(t)$ will be discontinuous at $t = 1$ if the limit of $\mathbf{V}(t)$ as t approaches 1 from the left fails to equal its value of $\mathbf{V}(1) = \mathbf{0}$; that is, for continuity at $t = 1$ we require

$$(2.3) \quad \lim_{t \rightarrow 1^-} \mathbf{V}(t) = \mathbf{V}(1) = \mathbf{0}.$$

Whether the limit condition (2.3) obtains depends on the notion of limit employed.

If we employ a weaker sense, admitted by Earman and Norton, then the limit condition (2.3) does obtain. That weaker sense merely requires that

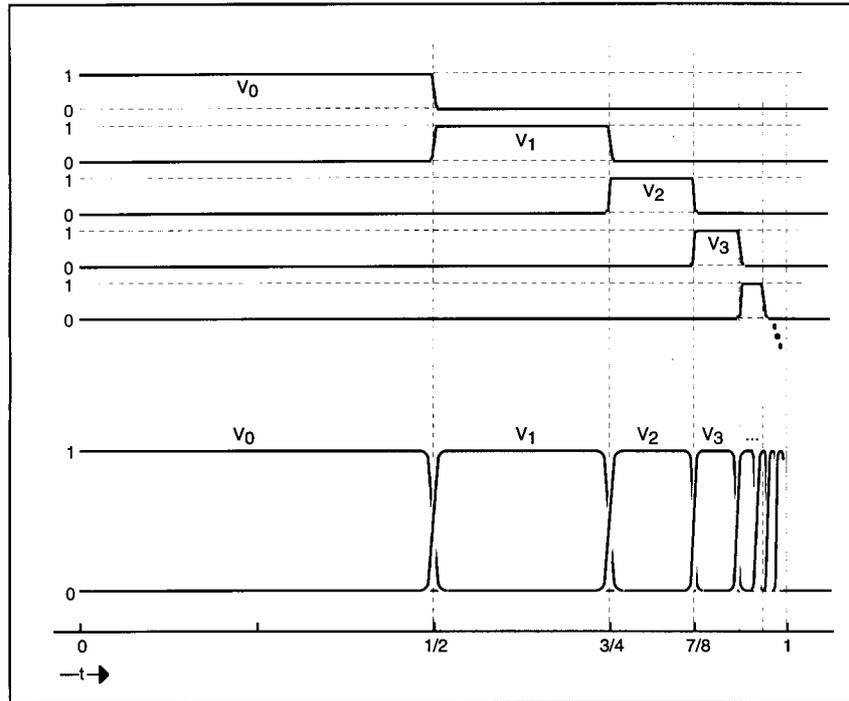


Figure 2. Components V_i as functions of t .

each component $V_i(t)$ individually approach its limiting value at $t = 1$, so that (2.3) requires only

$$(2.4) \quad \text{“component convergence”} : \lim_{t \rightarrow 1^-} |V_i(t) - V_i(1)| = 0$$

for all i .

This weaker condition will obtain for components $V_i(t)$ as shown in Figure 2. For each component V_i , there exists a time $t < 1$ after particle P_i 's collisions are completed at which $V_i(t) = 0$.³

Under a stronger sense of limit, required by Alper and Bridger, the limit condition (2.3) fails. The stronger sense requires that the normed difference of $\mathbf{V}(t)$ and $\mathbf{V}(1)$ vanish in the limit, so that (2.3) requires

$$(2.5) \quad \text{“convergence in the norm”} : \lim_{t \rightarrow 1^-} \|\mathbf{V}(t) - \mathbf{V}(1)\| = \mathbf{0}.$$

We can easily see that the limit condition fails in this stricter⁴ sense. The vector $\mathbf{V}(1) = \mathbf{0}$. But for any time $t < 1$, infinitely many particles are still to undergo collision so that we can always find a particle P_i that

will attain velocity $V_i = 1$ at some time between t and 1. At that time $\|\mathbf{V}(t) - \mathbf{V}(1)\| = 1$.

Component convergence merely requires that each component of the vector $\mathbf{V}(t)$ approaches its limit according to its own schedule. That is, for any $\epsilon > 0$, we can always find a time $t_i < 1$ for each $V_i(t)$ such that $V_i(t)$ is within ϵ of $V_i(1)$ for all $t_i < t < 1$. The higher numbered components may approach their limiting values later. Convergence in the norm, however, requires in addition that the components approach their limiting values with some degree of synchronization. That is, for any $\epsilon > 0$, there must be a *single* time T such that $\|\mathbf{V}(t) - \mathbf{V}(1)\|$ is within ϵ of 0 for all $T < t < 1$. By the definition of the norm, this entails that, for each i , $V_i(t)$ must be within ϵ of $V_i(1)$ for all $T < t < 1$.

4. THE SUPERTASK ST IS NEWTONIAN

Earman and Norton urge that the supertask ST be counted as Newtonian. The componentwise convergence of (2.4) is sufficient to ensure that each particle individually satisfies Newton's equations as expressed in (2.1) and (2.2). That is, by tracking the behavior of each individual particle (and even their pairwise interactions), we would have no reason to declare any violation of Newton's laws.

With each individual particle obeying Newton's laws (2.1) and (2.2), the behavior of the combined system becomes something to be discovered from appropriate calculations. Its properties are not to be stipulated by further legislation. In the case of ST, we discover that neither energy nor momentum is conserved and that, for the time reversed process, determinism fails.

We can add extra conditions if we like to preclude such pathologies. For example we could stipulate directly that, in addition to (2.1) and (2.2), energy is to be conserved. Or we could be somewhat indirect. Preserving energy conservation is the ultimate effect of the stronger convergence in the norm condition (2.5). That is, the squared norm $\|\mathbf{V}^2\|$ is twice the total kinetic energy of the particles. Energy conservation fails at $t = 1$ when this norm drops discontinuously with time to 0. The requirement of convergence in the norm (2.5) adds a requirement of continuity of the norm $\|\mathbf{V}\|$ itself to the component continuity of (2.4), thereby precluding the violation of energy conservation.⁵

Adding extra conditions such as energy conservation is not uninteresting. But it is not constitutive of Newtonian systems. Rather it leads us to investigate an important subset of all Newtonian systems, such as those that are deterministic and energy conserving. This particular subclass excludes

many interesting systems. For example, it excludes an infinite space populated with infinitely many colliding particles whose total energy is infinite. For such a system, conservation of total energy cannot be defined. But we may well be satisfied if energy is conserved in any finite region of space. If we are satisfied by that reduced form of energy conservation, why are we not also satisfied with a comparable reduction, energy conservation just for the individual collisions of ST? The subclass of deterministic, energy conserving systems would also fail to include recently investigated systems in which a small number of particles, interacting via a $1/r^2$ attractive force law, accelerate themselves off to spatial infinity in finite time, emptying space of particles. (See Mather and McGee, 1975; Earman and Norton, 1996, pp. 243–244.) As it turns out, the familiar Newtonian systems happen to be in the subclass of deterministic, energy conserving systems. But we should not turn this accident of the familiar into a necessity.

The decision as to which systems are Newtonian is not an empirical question. It cannot be decided by experiment in the way that we can decide whether electrons are spin half or spinless particles. It is a matter of labeling. However an incautious use of labels can do harm if it prevents us seeing important results. In this case, it is important to recognize that Newton's laws alone cannot assure us of determinism and energy conservation. If we build determinism and energy conservation into our definition of what it is to be Newtonian, we risk losing sight of this important result.

5. THE SUPERTASK ST IS NOT NEWTONIAN

Alper and Bridger believe that the supertask ST should not be called Newtonian. They agree that the component-wise solution to ST, $V_i(t) = 1$ for $t_i < t < t_{i+1}$ and $V_i(t) = 0$ otherwise, satisfies Newton's equations of motion for each $i > 0$ and for all t . (Here the t_i and t_{i+1} are the times that P_i is hit by P_{i-1} and hits P_{i+1} , respectively.) Assuming appropriately smoothed force functions, each $V_i(t)$, as well as each $X_i(t)$, becomes a differentiable function of t . For Earman and Norton, the existence of this well-behaved solution of Newton's equations is sufficient to characterize ST as Newtonian. However, although $X_i(t)$ and $V_i(t)$ are continuous functions of time, the energy is not continuous at $t = 1$: $E(t) = 1/2$ for $t < 1$ and $E(1) = 0$. This discontinuity means that energy is not conserved at $t = 1$.

Most physicists regard conservation of energy as a much more fundamental law than Newton's laws of motion (for example, Pais, 1986, 107–8). No physical system, when analyzed using the appropriate theory, be it classical, relativistic, or quantum mechanics, has been found to violate

the law of conservation of energy. The first law of thermodynamics, which states that the energy of an isolated system cannot change, is assumed to hold without exception. ST is an isolated system; it does not exchange either matter or energy with its surroundings. Therefore, it seems important to determine which feature of ST is responsible for its aberrant behavior.

ST exhibits none of the features that are the usual causes of energy non-conservation in dynamical systems governed by Newton's laws of motion. In ST, the particles are either at rest or move with finite constant velocity in a bounded region of space. The collisions are characterized as ordinary billiard ball collisions. There are no velocity dependent forces which are used to model energy losses due to friction. The system is autonomous, i.e., the equations of motion do not explicitly depend on time. Consequently, there is no transfer of energy between ST and some surrounding system. Finally, as the analysis of continuous systems and fields shows, even the presence of an infinite number of particles does not necessarily lead to energy nonconservation. Consider a vibrating one-dimensional elastic solid modeled by a chain of N equally spaced point masses connected by $N - 1$ massless springs. In the limit as N goes to infinity and the distance between particles goes to zero, the equation of motion for this system becomes the one-dimensional wave equation and energy is conserved.

Although infinite systems such as those described in the previous paragraph conserve energy, the limits involved in their definition are often subtle affairs. Consequently, we examine the limiting processes involved in defining the energy in ST.⁶

Let $E_N(t)$ be the energy of the finite system consisting of the particles P_0, P_1, \dots, P_N at time t . (This system is *not* a subsystem of ST. Instead, it is a new system used to describe an ordinary task in which P_N moves to the right unimpeded after it has been hit by P_{N-1} .) It seems natural to define the energy of the infinite system at time t as $\lim_{N \rightarrow \infty} E_N(t)$. For all N and for all t , including $t = 1$, $E_N(t) = 1/2$. Thus $\lim_{N \rightarrow \infty} E_N(t)$ is also equal to $1/2$ for all t . In particular, the energy at $t = 1$ is equal to $1/2$. By contrast, the energy, obtained from the component-wise analysis, is 0 for $t = 1$, since the velocity of each particle is 0 at $t = 1$. Clearly, unlike a continuous mechanical system or a field, ST cannot be expressed as a limit of a finite system as the number of particles goes to infinity.

It is possible to express $E(1)$, the energy of the completed supertask, as a limit as $t \rightarrow 1$. If $t < 1$, then we can write $E(t) = \frac{1}{2} V_i(t)^2$. In this expression, V_i is the velocity of the particle P_i . The subscript i is chosen so that the inequality $t_i < t \leq t_{i+1}$ is satisfied, and consequently, i is a function of t . As long as $t < 1$, $E(t) = 1/2$, so the limit as $t \rightarrow 1$ is also $1/2$. When $t = 1$, $E(1) = 0$, as we expect for the completed supertask.

Note, however, that the expression for E is quite unusual. The subscript i labeling the particle and the argument t are interdependent. Moreover, the closer t is to 1 the more accurately t must be known in order to determine i . As we show below, evaluating $E(t)$ on any time interval containing $t = 1$ requires that we know t *infinitely* accurately. Thus $E(t)$ is not well-defined on such an interval.

$E(t)$ is a step function. In order to evaluate it, we apply the simple rule: Determine whether t is less than 1; if it is, then $E(t) = 1/2$, else $E(t) = 0$. The simplicity of this rule hides a difficulty: we must know something about each of the infinitely many digits of the real number t . Consider the time given by the number whose decimal representation is computed to be $t^* = 0.9999999 \dots ??? \dots$. If t^* consists of all 9's, then $E(t^*) = 0$; however if any one of the digits of t^* differs from 9, then $E(t^*) = 1/2$. In other words, an arbitrarily small difference in t^* makes a macroscopic difference in $E(t^*)$.

Can we determine t^* with the kind of accuracy needed to determine $E(t^*)$? Clearly, t^* cannot be obtained from a scientific measurement because any measurement has only finite precision. No matter how many digits have been found by measurement to be 9, there can be no guarantee that they will all be 9. Moreover, even if there exists a theoretical procedure for calculating each digit of t^* , there may not exist a finite procedure for determining whether they are all equal to 9. Hence, it may be impossible to prove that $t^* < 1$. For such a t^* , the only way to determine whether $E(t^*) = 0$ or $1/2$ would be to complete the infinite task of separately examining each digit of t^* . Consequently, since $E(t)$ is defined only for those t for which we can determine whether $t < 1$, the energy function $E(t)$ is undefined in any interval containing 1.⁷

It is important for this argument to contrast the indeterminacy of $E(t)$ in a neighborhood of $t = 1$ with the situation in which a function $f(t)$ is known to be continuous. The real number $f(t)$ is determined if its value is known to within an arbitrary tolerance $\epsilon > 0$. By continuity, this condition can be satisfied if t is known to within some associated tolerance $\delta(\epsilon) > 0$. Thus, in order for a continuous function to be well-defined, we need only know the argument to arbitrary precision. We do not need infinite precision.

In view of the importance of conservation of energy for bounded isolated systems and further, the adverse implications of a discontinuous energy function, we believe that the global picture, involving the Hilbert space embeddings, is the appropriate model to use for analyzing ST. The Hilbert space norm provides enough structure to the spaces so that the notion of the distance between two velocity vectors can be defined. In this

global analysis, the equations of motion become singular at $t = 1$, so that Newton's equations can have no solution at this time. Consequently, the problems involving indeterminism and energy conservation dissolve. The inability to define the state of the system at $t = 1$, or equivalently, the singularity in Newton's equations at this time, provides the essential justification for refusing to call ST a Newtonian system.

6. EXPOSING A MATHEMATICAL LOOPHOLE

ST hides an essential ambiguity. The indeterminism claimed is not recovered directly from a mathematical description sufficiently detailed to demonstrate satisfaction of Newton's laws at all times. Instead it is recovered indirectly from meta-arguments that presume such a description is possible. For example, if ST is assumed to employ instantaneous collisions, the meta-argument could look like this. Energy and momentum conservation compel each individual collision to be resolved as in ST *if any resolution compatible with Newton's equations exists at all*.⁸

But to show that such a resolution exists requires that we make a careful mathematical model of the instantaneous and perfectly elastic collisions. This requires either distribution theory or positing the existence of strongly peaked force functions of unbounded strength, acting over arbitrarily short distances, and having an awkward accumulation point at $t = 1$. The simplicity of ST is slipping away.

Furthermore, if we allow meta-arguments to carry the day, the danger is that we cannot decide between competing meta-arguments. In our presentation of ST in Section 1, we allowed the end state of ST to be decided by a meta-argument that focused on the particles: for each particle $V_i(1) = 0$, and consequently, the energy of the system drops to zero at $t = 1$. How are we to arbitrate between this meta-argument and the following one that focuses on the energy interchanged in collisions, and gives a different conclusion: Let $E(0)$ be the initial energy of the system at $t = 0$, and C_i be the additional energy imparted to the system by the i th collision. Since all the collisions have been completed by time $t = 1$, $E(1) = E(0) + \sum_{i=1}^{\infty} C_i$. All of the C_i are zero, so that $E(1) = E(0)$ and energy is conserved. Perhaps this question can be resolved once the full mathematical descriptions are given, but that has yet to be done.

An example discussed elsewhere by one of us (Norton, 1999) shows that there is a system akin to ST which satisfies Newton's equations, violates determinism and energy conservation and for which an explicit mathematical demonstration of the violation is readily provided. The system consists of infinitely many unit masses, P_i , $i = 1, 2, 3, \dots$ in a row.

Each is connected by identical springs to its neighbors: P_1 to P_2 ; and P_i to P_{i-1} and P_{i+1} for $i > 1$. If the deviation of the i th mass from its equilibrium position is x_i , then Newton's laws applied to the i th mass ($i > 1$) becomes the equation of motion

$$\frac{d^2x_i}{dt^2} = k(x_{i+1} - x_i) - k(x_i - x_{i-1}),$$

where k is the Hooke's law spring constant. Thus if we consider the subsystem of the first n masses, for any n , we have n equations of motion (one for each mass) in $n + 1$ variables (the variables x_1, \dots, x_{n+1}). That is, for all of these subsystems, there is one more variable than equation so that the set of equations has no unique solution. We have indeterminism. Most of the admissible solutions violate energy conservation as well. This indeterminism depends essentially on there being infinitely many masses. If there were only finitely many, there would be a last mass P_N . The equation of motion governing it would have no term in x_{N+1} . We would have N equations in N variables and determinism would be restored.⁹

Indeterminism arises in ST because of an analogous structure in the governing equations. Consider the time reversed form of ST in which the particles are spontaneously excited by a disturbance that propagates from the accumulation point. If we consider the subsystem of the particles P_1, \dots, P_n , we will have n equations of motion in $n + 1$ variables, since the equation of motion for particle P_n must mention the position of particle P_{n+1} , with which it interacts by collision. For all these subsystems, we have one fewer equation than variable, and indeterminism results. Specifically, the speed of P_{n+1} at the time immediately before it collides with P_n can be chosen arbitrarily.

The point of this discussion has been to show that we are not arguing at cross purposes with one side setting meta-arguments against the other's demand for more complete mathematical description. The point at issue can be expressed in terms of mathematics.

For finite Newtonian systems, it is clear that there is one equation of motion for each one of the particles. Alper and Bridger require that in order for an infinite system to be called Newtonian, any finite subsystem used in the analysis of that system must also admit a description in which there is exactly one equation of motion for each particle. In their analysis of ST, the introduction of a norm in the Hilbert spaces serves as a proxy for the missing equation of motion they require. Earman and Norton believe there is no such requirement.

7. CONCLUSION

For an isolated dynamical system to earn the designation “Newtonian” is it necessary that it conserve energy? Alper and Bridger answer yes, arguing that the First Law of Thermodynamics trumps Newton’s laws of motion. Earman and Norton answer no, for otherwise we risk overlooking that Newton’s laws of motion are insufficient to ensure energy conservation and determinism.

We conclude with two points. First, irrespective of whether or not systems like ST or the example proposed by Norton should be called Newtonian, these systems are fascinating and are well worth (at least to us) careful study. Second, not all conceivable supertasks constructed with standard Newtonian components are Newtonian even in the weak sense favored by Earman and Norton. Both Alper and Bridger (1998, 366–7) and Earman and Norton (1998, 129–30) have described a system in which Newton’s laws are violated. In their example, if particle P_0 of Figure 1 were initially at some position X , with $X > 1$, and were to approach the remaining particles from *the right* with velocity $V = -1$, Newton’s laws will fail when P_0 arrives at $X = 1$.¹⁰

NOTES

- ¹ The precedent for this type of publication was set by Einstein and Ritz (1909).
- ² This sort of continuity need not be imposed after the fact on all supertasks. Norton (1999) describes an infinite system of masses connected by springs that spontaneously excites with all time dependencies remaining smooth. See also Grünbaum (1968) for a smoothing due to Richard Friedberg in an analogous case.
- ³ Here and below we assume that the regularization of the collisions has been effected by forces of sufficiently short range that, for all i , forces mediating the collision of P_{i-1} and P_i ceased to act before those mediating the collision of P_i and P_{i+1} begin to act.
- ⁴ Condition (2.5) is stronger than (2.4) in the sense that (2.5) entails (2.4) but not conversely. Should (2.4) fail for any value of i , then the non-zero $\lim_{t \rightarrow 1^-} |V_i(t) - V_i(1)|$ will contribute a positive term to the sum of squares on the left hand side of (2.5) ensuring the failure of condition (2.5). So (2.5) entails (2.4). However, as we show below, (2.5) forces continuity of the norm $\|\mathbf{V}(t)\|$ at $t = 1$ but (2.4) does not, so (2.4) cannot entail (2.5).
- ⁵ To see this, recall that (2.5) requires $\mathbf{0} = \lim_{t \rightarrow 1^-} \|\mathbf{V}(t) - \mathbf{V}(1)\|^2 = \lim_{t \rightarrow 1^-} \sum_{i=0}^{\infty} [V_i(t) - V_i(1)]^2 = \lim_{t \rightarrow 1^-} \sum_{i=0}^{\infty} ([V_i(t)]^2 - 2V_i(t)V_i(1) + [V_i(1)]^2) = \lim_{t \rightarrow 1^-} \sum_{i=0}^{\infty} ([V_i(t)]^2 - [V_i(1)]^2)$ since $\lim_{t \rightarrow 1^-} V_i(t) = V_i(1)$ for all i . Therefore $\lim_{t \rightarrow 1^-} \|\mathbf{V}(t)\| = \|\mathbf{V}(1)\|$.
- ⁶ In this discussion, we ignore the potential energy so that energy equals the sum of the kinetic energies of the particles. The potential energy can be ignored because during any collision, any change in potential energy is accompanied by an equal and opposite change of kinetic energy. Thus the occurrence of a collision, resulting in the interchange of potential and kinetic energy, has no effect on the value of the total energy.

⁷ An explicit example of such a t^* was suggested by John von Neumann. Let $t^* = 1 - G$, where the n th decimal place of G is 0 if $2n$ is a sum of two primes, and 1 otherwise. The value of G depends on the truth of the Goldbach conjecture which asserts that every even number is the sum of two primes. If Goldbach is true, then $G = 0$, if not, $G > 0$. Since the Goldbach conjecture is as yet unresolved, we do not know whether $t^* = 1$ or $t^* < 1$, even though t^* is computable to arbitrary accuracy.

⁸ If u and v are the velocities of the two particles, energy and momentum conservation require respectively $u^2 + v^2 = 1$ and $u + v = 1$. The two solutions are first $u = 1$ and $v = 0$ (the pre-collision velocities) and $u = 0$ and $v = 1$ (the post-collision velocities). Alternatively, one recovers the same result by requiring symmetry and energy conservation in the center of mass frame of reference and assuming that the system may be Galilean transformed.

⁹ We address a minor technicality. The indeterminism derives principally from the fact that there are fewer equations than variables in each subsystem. But this fact alone is not sufficient for indeterminism. To see this, consider infinitely many, real-valued variables $y_i(t)$, governed by $y_{i+1} = 2y_i/\sqrt{1 - y_i^2}$, where $i = 1, 2, 3, \dots$. The subsystem of the first n equations is deterministic. We can choose any function for $y_1(t)$ and read off remaining, compatible $y_2(t), \dots, y_n(t)$, as long as we ensure that $y_1(t)$ is sufficiently close to zero for all t , so that the remaining $y_2(t), \dots, y_n(t)$ are well defined. A choice of $y_1(t)$ that works for one subsystem might not for another. But, because of the factor of 2 in each equation, the only way *all* variables y_i can be defined in the infinite case is if we choose $y_1(t) = 0$ for all t ; thus determinism survives. By contrast, in the masses and springs the structure of the system is such that the arbitrariness of choice of $x_1(t)$ remains even when we consider the full, infinite set of equations.

¹⁰ It cannot halt or be deflected as a result of a collision with some particle P_i (for any i) since this collision is shielded by particle P_{i+1} . But neither can it continue unimpeded for it would have to pass through the higher numbered particles without deflection.

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