Persuasion with an Unknown Number of Signals

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Abstract

I study the effect of one-sided uncertainty of both the quantity of signals and their values in the context of the seller-receiver game where disclosure is truthful and verifiable. Unlike models where the number of signals is known, full disclosure is not an equilibrium outcome. This is due to senders with more information, some of which is not favorable, only revealing the good signals, allowing them to pool with senders that only have those signals. In addition there are equilibria in which the absence of disclosure is a positive probability outcome. I extend the model to allow the sender to pay a small fee to certify the quantity of signals which leads to more disclosure in equilibrium. Finally, I consider a multiple-sender environment and analyze fairness and efficiency. The receiver can make the game more fair by limiting the number of signals that each sender may transmit. Depending on the population of senders this may increase or decrease efficiency.

1 Introduction

Consider the following situation facing the manager of a firm. She seeks to maximize current capitalization and has a variety of information which she may share with the investing public. This includes data such as sales figures, reports from research and development on new products that the firm may introduce to the market or even information on the morale of the employees. An important characteristic of this data is that not only does the public not know the content of the reports, but it does not even know how many reports the manager has. The primary goal of this paper is to model this situation and examine how this uncertainty affects equilibrium outcomes.

Another relevant characteristic is the conflict in incentives between the investors and the firm. The investors wish to assess the value of the firm accurately while the manager wants for the investors to assess its value as highly as possible. Because the manager may selectively reveal information, this conflict leads to uncertainty on the part of the investors. As an example, suppose the manager of a firm releases favorable sales figures from Europe as well as Asia. The investor will not know whether the firm kept hidden a report on lackluster sales in North America or if this report was simply not available. Similarly, if the firm releases information on a future project that it expects to sell really well, the investors cannot be certain that existing reports on delays or high costs for other projects were not revealed.

There are several other real-world examples of this type of situation. In academia, letters of recommendation play an important role in applications both for graduate school and the market for junior economists. Some candidates come from large departments with several professors in their field, while others come from departments with only one or two such faculty members. If applicants could submit as many letters as they wished, then the hiring committee may not know if an applicant with few letters came from a small department or made an unfavorable impression on some faculty members at a larger department. Even with the standard three letters the same situation arises if an applicant has letters from one or two faculty members from their department and one or two from outside the department who are less familiar with the candidate. Other examples include the testimony of expert witnesses where the jury is not informed of how many experts each side talked to, media coverage of news events when some individuals or groups can control access and results found in research papers where a researcher may leave out unfavorable results.

When considering these environments, there are several questions that I will address. How much information will the sender reveal? Which information will the sender reveal? If the information quantity is exogenous, what are the effects of mandatory disclosure laws? How does the sender having the ability to certify how many signals she has affect outcomes? Beyond the question of how much information is revealed, there are issues of fairness and efficiency especially when one considers the natural extension of having several senders. Does the environment favor certain senders? Is there an efficiency loss? Finally, if a sender can verify her number of signals at some cost, how does that affect outcomes?

Despite the importance of these types of situations and how often they arise, the literature fails adequately to address these questions. There is a deep literature on disclosure, though it overwhelmingly addresses issues that arise when a sender has one signal of quality or a number of signals that is common knowledge. An example is the labeling of nutrition content of food. In early work in this area, such as Grossman and Hart [5], Milgrom [11], Milgrom and Roberts [12], they show that when the number of signals is common knowledge, the unique equilibrium outcome is full disclosure. However, in practice, when there are no mandatory disclosure laws less than full disclosure is common.

Due to this disparity, a variety of papers have been written that show that there are equilibria where there is less than full disclosure. Reasons given include disclosure being costly (Jovanovich [8], Grossman and Hart [5]), that competing firms may wish not to reveal some signal values to decrease price competition (Board [1]), the existence of a second-stage where a sender transmits a noisy signal based on her true quality (Harbaugh and To [7]) or that firms may be uninformed of their quality (Dye [3], Jung and Kwon [9], Matthews and Postlewaite [10]).

Of this early work, the Dye and the Jung and Kwon papers are closest to mine in that the amount of information that the firm has is unknown to the receiver. In each of these papers, a sender with a small signal will pool with the uninformed senders and disclose nothing. My paper is both an extension of this work and a notable departure from it. Firstly, I depart by considering the case where the sender has one signal or two signals. That is, the sender is never fully uninformed. I also extend my model to consider the possibility that a sender is uninformed (has no signals), has one signal or has two signals. Each of these non-trivially alters the equilibrium set.

Recent work from Dziuda [4] allows for the receiver not to know how many signals the sender has. In her case, there are two alternatives and several signals of quality. Each signal is simply an indicator of which alternative is better. The quality of each alternative is then the proportion of signals favoring it. The number of signals is itself a random draw only known to the sender. The sender chooses how many of each type of signal to reveal to the receiver. With this information, the receiver chooses one of the two alternatives. Her model allows for the sender to be biased in either direction and possibly neutral. It describes certain situations quite well. For example, suppose a cancer patient is considering two treatment options. He doesn't know all of the factors that should be considered and because they can involve complicated medical terminology and concepts, he also cannot fully interpret information on these factors. He will likely make his decision based on which treatment he feels has more favorable aspects.

There is, however, one limitation that means that her paper does not adequately address the motivating examples I've given above: the signals in her model are binary. Each signal gives some evidence which indicates which of the two alternatives is better, but individually they cannot say how much better. For this to describe some real-world situation, either the signals actually must be binary or the receiver not be able to interpret them. Note that in the patient example in the previous paragraph the latter was true. Even examples like this will often suffer from the receiver having the ability to interpret at least a few of the signals themselves. My motivating examples all involve people who are able to understand the data, but are uninformed. In these cases, my design more accurately models these situations.

The difference between our models leads to a disparity in results. She finds that a biased sender will disclose all favorable information and some unfavorable information. This holds no matter the number of signals. While there may be multiple equilibria in my model, in all equilibria if the sender has multiple signals and one of them is small enough, then she will not disclose it.

My paper makes several contributions to this literature. My model is able to analyze a variety of situations that those of the existing literature can not. The solutions to the game are themselves important. The first important result is that for all distributions and probabilities, with positive probability the sender will not disclose fully. I also show that with many, but not all, distributions there are equilibria where with positive probability the sender discloses nothing. This is interesting because it is true despite it being common knowledge that the sender has at least one signal. Equilibria of this type also satisfy such criteria as the intuitive criterion. These two results provide some justification for mandatory disclosure laws. The uncertainty present also allows for me to consider certification of signal quantity, which is an issue that can not be captured by these other models. In addition to these contributions, with my model I am able to assess fairness and efficiency in multiple-sender games. My model allows for the realistic possibility that two senders be equivalent in terms of their quality but have different amounts of information. In all of the papers where the number of signals is common knowledge, this is not possible. Senders with the same quality always get the same outcome because there is full disclosure.

The paper is organized as follows. In the next section I will analyze the game where the sender has either one or two signals. I find all equilibria that involve cutoff strategies. No matter the probability that the sender has one or two signals or the distribution of those signals, there is always a unique equilibrium in which the sender discloses fully if she has a single signal. For some distributions and probabilities there are also equilibria where the sender with one signal discloses if its value is above some cutoff and does not below this cutoff. There are robustness issues with these equilibria, which I discuss in Section 2.3.2. In Section 3, I allow for the sender to have zero, one or two signals. This causes the sender to play equilibria that are similar in nature to those of section 2.3 where the sender sets a positive cutoff. In Section 4 I alter the game to allow the sender to pay a small fee to have an outside agent certify the number of signals that she has. Allowing for this leads to more disclosure in equilibrium. In Section 5, I extend the game to consider multiple senders. This allows me to consider efficiency and fairness. I find that the game is unfair and inefficient. A social planner could improve fairness by restricting the strategy space of the sender. Depending on the distribution and probabilities, this may increase or decrease efficiency. Section 6 discusses several extensions and concludes.

2 The Sender-Receiver Game with One or Two Signals

2.1 The Model

The general structure of the models is as follows. The risk-neutral female sender has either one or two signals of quality. One can think of the sender as a firm selling a good and the signals as indicators of its quality. If there is only one signal then its value determines her quality. If there are two signals, then her quality is their average. She reveals both, one, or none of her signal(s) to the risk-neutral receiver. With the information that the sender has revealed, the risk-neutral male receiver estimates the quality of the product. The sender wants the receiver to give a high estimate while the receiver wants to guess accurately.

Here is the model more formally. The sender has either one or two signals of quality. Letting s denote the number of signals that the sender has, the receiver believes ex-ante that $s = 1$ with probability $p \in (0, 1)$ and $s = 2$ with probability $1 - p$. Furthermore, if $s = 1$ then the receiver believes that the signal is drawn from some continuous distribution F with full support on the unit interval. If $s = 2$, then he believes that the signals are drawn from joint density $g(\cdot, \cdot)$, which is also continuous and has full support on the unit square. It is further assumed that g is symmetric, so $g(x, y) = g(y, x)$ for all x and y in the unit interval. If the sender has one signal then her quality is the value of this signal. If she has two then her quality is the average of her signals:

$$q = \begin{cases} x & \text{if } s = 1 \\ \frac{1}{2}(x + y) & \text{if } s = 2 \end{cases}$$

where x and y are value(s) of the signal or signals and q is the actual quality of the sender.

The sender may reveal both, one, or none of her signal(s) to the receiver. This disclosure is truthful and verifiable. After getting the signals, the receiver estimates the quality of the sender. If the actual quality of the sender is q and the receiver's guess is $\hat{q}$ then their payoffs are

$$U_S(q,\hat{q}) = \hat{q}$$
$$U_R(q,\hat{q}) = -(q-\hat{q})^2$$

I am using a quadratic-loss payoff function for the receiver. This should not fundamentally alter the predictions of the model. I use it for convenience because with this payoff function the receiver will optimally set $\hat{q}$ equal to the expected quality of the sender, conditional on what the sender has disclosed. An outcome equivalent design would be to have the receiver's payoff be the quality of the good net of the price which the sender would set in a second stage.

Note that the quality of the good does not directly affect the payoff of the sender. This is similar to the standard buyer-seller assumption that the good is worthless to the seller if not sold. Also important is the conflict of incentives. While the sender only wants the receiver to set a high estimate, the receiver wants to be accurate.

The state space is $[0,1] \cup [0,1] \times [0,1]$. That is, the sender can have one signal or two and all signals are in the unit interval. However, if the sender has two signals there is no reason to think of them as ordered. Therefore, in order for the state space to be accurately described I restrict the strategy space so that the ordering doesn't matter. With that in mind, a strategy is a function $\sigma : [0,1] \cup [0,1] \times [0,1] \to \emptyset \cup [0,1] \cup [0,1] \times [0,1]$ that satisfies the following properties

1. For all $x \in [0,1]$, $\sigma(x) \in \emptyset \cup \{x\}$.
2. For all $(x,y) \in [0,1] \times [0,1]$, $\sigma(x,y) \in \emptyset \cup \{(x,y), x, y\}$.
3. For all $(x,y) \in [0,1] \times [0,1]$, $\sigma(x,y) = \sigma(y,x)$.

The first two properties state that the sender may disclose nothing, one, or both of her signal(s) but may not claim to have a signal that she doesn't have. The third property states that her strategy must be indifferent to the ordering of her signals allowing me to think of two signals as an ordered pair without affecting the outcome. This will be convenient later as it allows for a graphical interpretation of strategies.

Due to the uncertainty present in this model, I will use Bayesian perfection as the equilibrium concept. In a Perfect Bayesian Equilibrium, the receiver forms beliefs over the possible signal holdings. These beliefs must be consistent with the strategy of the sender, updating using Bayes' rule wherever possible. In addition, the strategy of the sender must be optimal given the strategy of the receiver and the strategy of the receiver must be optimal given his beliefs.

Let $\sigma^{-1}(\emptyset)$, $\sigma^{-1}(x)$, $\sigma(x,y)$ be the set of signals and pairs of signals such that the sender discloses nothing, only x, and both x and y respectively. If the sender is playing a pure strategy, the on-equilibrium beliefs are the Bayesian updated probability that the sender has a given signal or signal pair taking into account the probability distributions, her strategy, and what she has revealed. For example, suppose she has revealed nothing. Then the weight that should be given to her having a single signal with value $\frac{1}{2}$ is

$$P\left(\frac{1}{2} \mid \emptyset\right) = \frac{1_{\sigma^{-1}(\emptyset)}\left(\frac{1}{2}\right) P\left(\frac{1}{2} \mid s=1\right) P\left(s=1\right)}{\left(\begin{array}{c}\int_0^1 1_{\sigma^{-1}(\emptyset)}\left(x\right) P\left(x \mid s=1\right) P\left(s=1\right) dx \\ +\int_0^1 \int_0^1 1_{\sigma^{-1}(\emptyset)}\left(\left(x,y\right)\right) P\left(\left(x,y\right) \mid s=2\right) P\left(s=2\right) dxdy\end{array}\right)}$$

$$= \frac{1_{\sigma^{-1}(\emptyset)}\left(\frac{1}{2}\right) f\left(\frac{1}{2}\right) p}{\left(\begin{array}{c}\int_0^1 1_{\sigma^{-1}(\emptyset)}\left(x\right) f\left(x\right) p dx \\ +\int_0^1 \int_0^1 1_{\sigma^{-1}(\emptyset)}\left(\left(x,y\right)\right) g\left(x,y\right)\left(1-p\right) dxdy\end{array}\right)}$$

where $1_{\sigma^{-1}(\emptyset)}(x)$ $(1_{\sigma^{-1}(\emptyset)}((x,y)))$ is one if the sender's strategy is for her to disclose nothing if she has x (x and y) and zero if she would disclose x (x or y). Off the equilibrium path Bayesian updating is not possible because the denominator is zero. In this case the receiver may apply any distribution he wishes to $[0,1] \cup [0,1] \times [0,1]$.

As stated above, in the second stage it is optimal for the receiver to set $\hat{q}$ equal to the expected quality of the sender, conditional on his beliefs. Therefore, an equilibrium is essentially a strategy for the sender that maximizes the expected quality of the product from the receivers perspective, for each possible number and value of signals.

2.2 Characterization of Cutoff Strategy Equilibrium Outcomes

In this section, I characterize all equilibria involving cutoff strategies. Broadly speaking, a cutoff strategy is one where the sender discloses if she has one signal and it is above a certain cutoff, reveals the larger of two signals if it is above some value and the smaller if it is large enough relative to the value of the revealed signal. Finally if the larger signal is below the relevant cutoff, the sender discloses both of her signals if their average is above a cutoff. If the sender has one signal then she will reveal it if it large enough that she will benefit from revealing. With two signals, her thought process is to first consider whether she is better off revealing only the larger signal than nothing at all. If this is the case then she decides whether it is better to reveal both signals or only the larger. If it is better to reveal nothing than the larger signal, she determines whether she can improve by revealing both. One can see that this line of thought will always lead her to the optimal action so long as her assumptions about the response of the receiver to her disclosure are accurate, and it is never strictly better to disclose only the smaller of two signals.

I am restricting my attention to equilibria of this type because other equilibria seem implausible in a real-world setting. If the one-signal part of the sender's strategy does not satisfy the cutoff assumptions then in equilibrium she must find it optimal to not disclose if she has some signal but reveal if she has some signal that is smaller. Not only would such an equilibrium strategy be counter-intuitive and complicated, but it would not do well in a richer environment where, for example, the sender is not sure which of several equilibrium beliefs and strategies the receiver is adopting. The same holds for strategies where the sender either reveals only the smaller of two signals or there is a similar lack of monotonicity when only looking at the larger signal.

It is important to note that while I am only finding equilibria where the sender plays a cutoff strategy, I am not restricting the strategy space to such strategies. In other words, in each equilibrium the strategy of the sender yields a payoff that is at least as good as any other strategy, including strategies that are not cutoff.

In this section, I will give a full characterization of the set of cutoff-strategy equilibria. After formally defining a cutoff strategy, I divide the set of equilibria into three types. This distinction is based on what the sender does when she has one signal. In the first type of equilibrium, the sender will always disclose if she has only one signal that is positive. This equilibrium, which always exists and is unique, is the most intuitive. In the second type, the sender with one signal never discloses it. Finally, I consider the in-between case where if the sender has one signal she discloses it if is above some positive value and does not if it is below.

2.2.1 Description and Notation

By $E(q \mid x)$ I refer to the expected quality of the sender conditional on her strategy, the prior, and that she has disclosed a single signal with value x. This expectation, like all others, is from the perspective of the receiver. Similarly, $E(q \mid x, y)$ is the expected quality of the sender when she discloses signals with values equal to x and y respectively. Note that if the sender has disclosed x and y then she must have precisely these signals and hence her quality level must be $\frac{1}{2}(x+y)$. Finally, I stray from the standard use of the empty set and let $E(q \mid \emptyset)$ denote the expected quality of the sender when she discloses no signals. While technically the beliefs of the receiver are a probability distribution over $[0,1] \cup [0,1] \times [0,1]$, I will not state them in this way for outcomes that occur in equilibrium. Instead I will describe the set of signals and signal pairs for which the receiver believes that the sender would take the given action. The probability distribution for the beliefs can then be determined using Bayesian updating.

A cutoff strategy in this model is a bit more complicated than a standard disclosure model because it must describe what the sender will do when she has two signals. A cutoff strategy has the following four properties.

P1 There is a value c_1 such that if the sender has a single signal and its value is greater than c_1 then she discloses it. If the sender has a single signal and it is less than c_1 then she does not disclose.

P2 There is a value c_2 such that if the sender has two signals, the larger of which is greater than c_2, then she will disclose at least her largest signal.

P3 Suppose the sender has two signals x and y with $x \geq y$. There is a function $\gamma : [c_2, 1] \rightarrow [0,1]$ such that if $x > c_2$, so that the sender discloses x, then the sender discloses y if $y > \gamma(x)$ and does not if $y < \gamma(x)$.

P4 Suppose the sender has two signals x and y with $x \geq y$. There is a value k such that if $x < c_2$ then the sender discloses both x and y if $\frac{1}{2}(x+y) > k$ and discloses neither if $\frac{1}{2}(x+y) < k$.

The first property is the standard cutoff for the one signal game. The second property is similar, stating that if the larger of two signals is above a cutoff then it should be revealed. The third property states that if the sender's larger signal is above this cutoff, then there is a cutoff point for the disclosure of the smaller signal. This cutoff depends on the value of the larger signal. Finally, the fourth property allows for the possibility that it may be better off for the sender to reveal both signals even if it is suboptimal to reveal only the larger signal.

An important aspect of cutoff strategies is that they are pure strategies except possibly at the cutoffs themselves. Because the sender has a signal or pair of signals at the cutoff with probability zero, this does not affect anything. In equilibrium, At the cutoffs the sender is indifferent between the given actions. For simplicity, I will assume that the sender always takes one of the optimal actions at the cutoffs and is therefore playing a pure strategy.

2.2.2 When the One-Signal sender Always Discloses

Before characterizing all cutoff equilibria where the sender always discloses if she has one signal, I will give a clarifying example. Suppose the sender has one or two signals with equal probability. If she has one signal, then it is uniformly distributed. If she has two signals, then these draws are independent and uniformly distributed. That is, $f(x) = g(x,y) = g_{x|y}(x) = 1$, $F(x) = G_{x|y}(x) = x$, and $p = 0.5$. Let the sender always disclose if she has one signal and always disclose the larger of two signals if it is positive. If she has two signals and both are zero, she will disclose nothing. To fully describe the equilibrium, I would only need to specify the values over which the sender discloses her smaller signal, give the expected quality of the good for every possible signal and signal pair that the receiver can see, and specify off-equilibrium beliefs for those signals and pairs that do not occur in equilibrium. Note that with this strategy the sender should always disclose at least one signal, unless she has two signals that are zero. Therefore, if the sender discloses nothing the receiver should place full weight on her having two signals both of which are zero.

To determine the cutoff function for the smaller signal, suppose the sender's larger signal is some arbitrary $x > 0$, which she discloses. Suppose further that if her smaller signal, y, is greater than $\gamma(x)$ then she will disclose it as well but will disclose only x if $y < \gamma(x)$. If she discloses only x then, from the receiver's perspective, the probability that she only has one signal is given by

$$\begin{aligned} P(s=1 \mid x) &= \frac{P(x \mid s=1) P(s=1)}{P(x \mid s=1) P(s=1) + P(x \mid s=2) P(s=2)} \\ &= \frac{pf(x)}{pf(x) + 2(1-p) P((x,y) \mid y < \gamma)} = \frac{1}{1+2\gamma}. \end{aligned}$$

Similarly, the probability that the sender has two signals is given by

$$P(s=2 \mid x) = \frac{2\gamma}{1+2\gamma}$$

Her expected quality is given by

$$\begin{aligned} E(q \mid x) &= \left(\frac{1}{1+2\gamma}\right)x + \left(\frac{2\gamma}{1+2\gamma}\right)\left(\frac{1}{2}\right)(x + E(y \mid y < \gamma)) \\ &= \frac{x + \gamma\left(x + \frac{1}{2}\gamma\right)}{1+2\gamma} = \frac{2x(1+\gamma)+\gamma}{2(1+2\gamma)}. \end{aligned}$$

The sender will disclose the lower signal if it is optimal for her to do so. To determine when this is the case note that if she discloses two signals, then her expected quality is simply the average of these two signals. For it to be incentive compatible, the sender must be indifferent between revealing both or just the larger, when her smaller signal is at the cutoff. That is, it must be that

$$\begin{aligned} E(q \mid x) &= E(q \mid x, \gamma) \\ \frac{2x(1+\gamma)+\gamma}{2(1+2\gamma)} &= \frac{1}{2}(x+\gamma). \end{aligned}$$

Solving for γ gives

$$\gamma(x) = \frac{1}{2}\left(\sqrt{1+4x} - 1\right).$$

In summary, the sender will always disclose if she has a single signal. She will always disclose the larger of two signals if it is positive. If her smaller signal is larger than γ evaluated at the value of her larger signal, then she discloses it as well. If it is smaller than this value then she does not disclose it. If she discloses signals x and y then her expected quality is $\frac{1}{2}(x+y)$. If she discloses only x then her expected quality is $\frac{1}{2}(x+\gamma(x))$. Graphically, this is shown in figure 1. In the yellow portion of the graph, the sender discloses fully revealing either her only signal or both of her signals if she has two. In the green section she discloses only the larger signal.

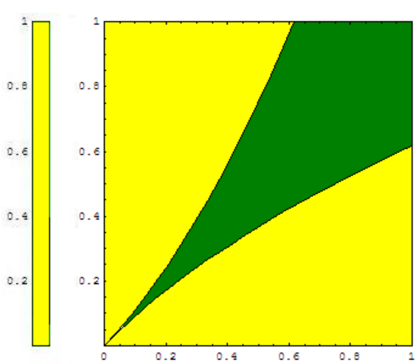


Figure 1: In the yellow region the sender discloses one signal. In the green region she discloses two signals.

This equilibrium has several properties. Firstly, the sender always discloses at least one signal, so there is always some disclosure. Like the unraveling result, the receiver believes that the sender is of quality 0 if she discloses nothing. While there is some disclosure, there is not full disclosure. With probability .348, half of the green area in Figure 1, the sender does not disclose fully. When the sender has two signals that are close in value then she will disclose both. It is only when she has two signals that are more spread out that she opts to disclose only one signal, pooling with that type of sender.

All equilibria in which the sender always discloses when she has one signal have similar properties. In order to show that, it is first necessary to show that a sender with two signals, at least one of which is positive, will always disclose at least one positive signal. This is an immediate corollary of the following result, which is that if the sender discloses nothing or a single signal of value 0 then her payoff is zero.

Lemma 1 *If σ is an equilibrium pure strategy such that $\sigma(x) = x$ for all $x > 0$ then $E(q \mid 0) = E(q \mid \emptyset) = 0$.*

Proof. Suppose $\sigma(0,0) = 0$. Then if $E(q \mid 0) > 0$ there must be some $\frac{1}{2}y > E(q \mid 0)$ such that $\sigma(0,y) = 0$. This contradicts optimality and therefore $E(q \mid 0) = 0$. Similarly, if $E(q \mid \emptyset) > 0$ the sender with $(0,0)$ could improve by disclosing nothing and hence we must have $E(q \mid \emptyset) = 0$.

Suppose $\sigma(0,0) = (0,0)$. Then if $E(q \mid 0) > 0$ ($E(q \mid \emptyset) > 0$) the sender could improve by revealing 0 (nothing), a contradiction.

Suppose $\sigma(0,0) = \emptyset$. Then if $E(q \mid \emptyset) > 0$ there is an $(x,y) \in [0,1] \times [0,1]$ such that $\frac{1}{2}(x+y) > E(q \mid \emptyset)$ and $\sigma(x,y) = \emptyset$, a contradiction since the sender would get $\frac{1}{2}(x+y)$ from revealing both x and y. Hence, the sender gets a payoff of 0 from disclosing nothing. If $E(q \mid 0) > 0$ then the sender with $(0,0)$ could improve by disclosing one of her signals. ■

Corollary 1 *If σ is an equilibrium pure strategy such that $\sigma(x) = x$ for all $x > 0$ then for all $(x,y) \in (0,1] \times (0,1]$, $\sigma(x,y) \in \{x, y, (x,y)\}$. If $\sigma(x,y) \in \{x,y\}$ then $\sigma(x,y) > 0$.*

Proof. Because the disclosure of at least one signal that is positive will always give the sender a positive payoff, this is an immediate result of the previous lemma. ■

So far in my equilibrium description, the sender always reveals if she has one signal and always discloses at least one of two signals if at least one of them is positive. I will now discuss with which pairs of signals she will reveal both. Suppose the sender reveals signal x and is then considering also disclosing y. If the sender discloses both signals then she gets a payoff of $\frac{1}{2}(x+y)$. By definition, if she discloses only x she will get $E(q \mid x)$, which will be determined shortly. It is clearly correct for her to disclose y in addition to x if

$$E(q \mid x) < \frac{1}{2}(x+y).$$

Similarly, it is correct to disclose x and not y if

$$E(q \mid x) > \frac{1}{2}(x+y).$$

This means that for every x there is some cutoff, which I'll call γ, such that if $y > \gamma$ then she will disclose y but if $y < \gamma$ then she will not. Because the cutoff could change for different values of x, one could think of the cutoff as a function that sends a point x to another point in $[0,1]$ above which the sender would disclose both signals and below which she would disclose only x, being indifferent at this value.

To find this function $\gamma : [0,1] \to [0,1]$ I will let arbitrary $x \in [0,1]$ be given and calculate $E(q \mid x)$ when the sender does not disclose a second signal if it is less than γ. I will then set this equal to $\frac{1}{2}(x+\gamma)$, the payoff to a sender who discloses both x and γ. This will give the indifference point. We have

$$\begin{aligned}
E(q \mid x) &= P(s=1 \mid x)\, x + P(s=2 \mid x) \frac{1}{2}(x + E(y \mid y<\gamma)) \\
&= \frac{P(x \mid s=1) P(s=1)\, x + P(x \mid s=2) P(s=2) \frac{1}{2}(x + E(y \mid y<\gamma))}{P(x \mid s=1) P(s=1) + P(x \mid s=2) P(s=2)} \\
&= \frac{f(x)\, px + (1-p) P(x, y<\gamma)(x + E(y \mid y<\gamma))}{pf(x) + 2(1-p) P(x=x, y<\gamma)} \\
&= \frac{px + (1-p) P(y<\gamma)(x + E(y \mid y<\gamma))}{p + 2(1-p) P(y<\gamma)} \\
&= \frac{px + (1-p) G_{y|x}(\gamma)(x + E(y \mid y<\gamma))}{p + 2(1-p) G_{y|x}(\gamma)}
\end{aligned}$$

Setting this equal to $\frac{1}{2}(x+\gamma)$ gives

$$\frac{1}{2}(x+\gamma) = \frac{px + (1-p) G_{y|x}(\gamma)(x + E(y \mid y<\gamma))}{p + 2(1-p) G_{y|x}(\gamma)}. \tag{1}$$

Examining the equation, one can see that for most values of x there are two possible values for γ that would be valid. For $\gamma(\cdot)$ to make sense in the model, it should map $[0,1]$ into $[0,1]$. Fortunately, for every $x \in [0,1]$ there is some $\gamma \in [0,1]$ such that 1 holds. Moreover, within the unit interval, for any given x, this γ is unique.

Theorem 1 *For every $x \in [0,1]$ there is a unique $\gamma \in [0,1]$ such that 1 holds.*

Proof. I will first show existence and then show uniqueness. ■
Proof. Let $x \in [0,1]$ be given and

$$R(\gamma) = \frac{1}{2}(x+\gamma)\left(p + 2(1-p) G_{y|x}(\gamma)\right) - \left(px + (1-p) G_{y|x}(\gamma)(x + E(y \mid y<\gamma))\right)$$

Because $p + 2(1-p) G_{y|x}(\gamma)$ must be positive for all $\gamma \in [0,1]$, the claim is proven if it is shown that there is some $\gamma \in [0,1]$ such that $R(\gamma) = 0$. By the intermediate value theorem, it suffices to show that $R(0) \leq 0$ and $R(1) \geq 0$.

$$\begin{aligned}
R(0) &= \frac{1}{2}xp - xp = -\frac{1}{2}xp \qquad (2) \\
R(1) &= \frac{1}{2}(x+1)(p + 2(1-p)) - (xp + (1-p)(1 + E(y))) \\
&= \frac{1}{2}(1-x)\, p + (1 - E(y))(1-p) \qquad (3)
\end{aligned}$$

Because $p \in (0,1)$ and $E(y) < 1$ we must have $R(0) < 0$ and $R(1) > 0$ so existence has been proven.

For uniqueness, let $x \in [0,1]$ be given and suppose that γ and γ' each satisfy 1. Suppose with no loss of generality that $\gamma \geq \gamma'$. Rewriting 1, we have

$$\begin{aligned}(x+\gamma)\left(p+2(1-p)G_{y|x}(\gamma)\right) &= 2px+2(1-p)G_{y|x}(\gamma)(x+E(y\mid y<\gamma))\\(x+\gamma')\left(p+2(1-p)G_{y|x}(\gamma')\right) &= 2px+2(1-p)G_{y|x}(\gamma')(x+E(y\mid y<\gamma')).\end{aligned}$$

Subtracting the second equation from the first gives

$$\begin{aligned}&2(1-p)\left(x\left(G_{y|x}(\gamma)-G_{y|x}(\gamma')\right)+\gamma F(\gamma')-\gamma' G_{y|x}(\gamma')\right)+p(\gamma-\gamma')=\\&\quad 2(1-p)\left(x\left(G_{y|x}(\gamma)-G_{y|x}(\gamma')\right)+G_{y|x}(\gamma)E(y\mid y<\gamma)-G_{y|x}(\gamma')E(y\mid y<\gamma')\right)\end{aligned}$$

or

$$2(1-p)\left(\left(\gamma G_{y|x}(\gamma')-\gamma' G_{y|x}(\gamma')\right)-\left(G_{y|x}(\gamma)E(y\mid y<\gamma)-G_{y|x}(\gamma')E(y\mid y<\gamma')\right)\right)+p(\gamma-\gamma')=0.$$

By assumption, $\gamma-\gamma' \geq 0$ so it must be the case that

$$\gamma G_{y|x}(\gamma')-\gamma' G_{y|x}(\gamma')-\left(G_{y|x}(\gamma)E(y\mid y<\gamma)-G_{y|x}(\gamma')E(y\mid y<\gamma')\right)\leq 0. \qquad (4)$$

However, note that

$$\begin{aligned}G_{y|x}(\gamma)E(y\mid y<\gamma) &= \int_0^{\gamma} z\cdot g_{y|x}(z)\,dz\\ \text{and } G_{y|x}(\gamma)E(y\mid y<\gamma') &= \int_0^{\gamma'} z\cdot g_{y|x}(z)\,dz\end{aligned}$$

Therefore,

$$\begin{aligned}\left(G_{y|x}(\gamma)E(y\mid y<\gamma)-G_{y|x}(\gamma')E(y\mid y<\gamma')\right) &= \int_0^{\gamma} z\cdot g_x(z)\,dz-\int_0^{\gamma'} z\cdot g_x(z)\,dz\\ &= \int_{\gamma'}^{\gamma} z\cdot g_x(z)\,dz.\end{aligned}$$

Similarly, by definition

$$\begin{aligned}G_{y|x}(\gamma) &= \int_0^{\gamma} g_x(z)\,dz\\ \text{and } G_{y|x}(\gamma') &= \int_0^{\gamma'} g_x(z)\,dz.\end{aligned}$$

So

$$\begin{aligned}\gamma G_{y|x}(\gamma') - \gamma' G_{y|x}(\gamma') &= \gamma \int_0^{\gamma} g_x(z)\,dz - \gamma' \int_0^{\gamma'} g_x(z)\,dz \\ &= \gamma \left(\int_0^{\gamma'} g_x(z)\,dz + \int_{\gamma'}^{\gamma} g_x(z)\,dz \right) - \gamma' \int_0^{\gamma'} g_x(z)\,dz \\ &= (\gamma - \gamma') \int_0^{\gamma'} g_x(z)\,dz + \gamma \int_{\gamma'}^{\gamma} g_x(z)\,dz.\end{aligned}$$

Therefore, the left hand side of 4 may be rewritten

$$\begin{aligned}&\gamma G_{y|x}(\gamma') - \gamma' G_{y|x}(\gamma') - \left(G_{y|x}(\gamma) E(y \mid y < \gamma) - G_{y|x}(\gamma') E(y \mid y < \gamma')\right) \\ &\qquad = (\gamma - \gamma') \int_0^{\gamma'} g_x(z)\,dz + \int_{\gamma'}^{\gamma} (\gamma - z)\, g_x(z)\,dz\end{aligned}$$

The first term is positive by assumption unless $\gamma = \gamma'$, in which case it is zero. If $\gamma > \gamma'$, then the second term must be positive which is a contradiction. Therefore, $\gamma = \gamma'$ and uniqueness has been proven. ∎

In discussing the equilibrium, I stated that the sender always reveals at least one signal if she has two. I did not state which she would reveal. Intuitively, it should be the case that the sender would always reveal the larger of her two signals and sometimes reveal the smaller based on its value. That holds for this model. It is guaranteed by the fact that the γ defined implicitly by 1 is increasing. The reason for this is that the payoff to the sender who reveals a single signal x is $\frac{1}{2}(x + \gamma(x))$. The only way this could be non-increasing in x is for γ to be decreasing in x. That is not the case.

Theorem 2 *Let $\gamma : [0,1] \to [0,1]$ be the function implicitly defined by 1. γ is increasing.*

Proof. This proof will be done in two steps. First, I will show that γ must be one-to-one. Then I will use this to show that it must be increasing.

Let x and x' be given. Suppose without loss of generality that $x \geq x'$. Let $\gamma = \gamma(x) = \gamma(x')$. Then rewriting 1, we have

$$\begin{aligned}(x + \gamma)(p + 2(1-p) G_x(\gamma)) &= 2px + 2(1-p) G_x(\gamma)(x + E(y \mid y < \gamma)) \\ (x' + \gamma)(p + 2(1-p) G_x(\gamma)) &= 2px' + 2(1-p) G_x(\gamma)(x' + E(y \mid y < \gamma)).\end{aligned}$$

Subtracting the second equation from the first and then subtracting the right side from the left we have

$$p(x' - x) = 0.$$

This is equal to zero only when $x' = x$ and therefore $\gamma : [0,1] \to [0,1]$ must be one-to-one.

To see that this implies that γ must be increasing, consider $\gamma(0)$ and $\gamma(x)$ for some $x \in (0,1)$. For $\gamma(0)$ we have

$$\frac{1}{2}(0 + \gamma) = \frac{p \cdot 0 + (1-p) G_x(\gamma)(0 + E(y \mid y < \gamma))}{p + 2(1-p) G_x(\gamma)}$$

or

$$\gamma = \frac{2(1-p)\,G_x(\gamma)\,E(y \mid y < \gamma)}{p + 2(1-p)\,G_x(\gamma)}$$

Because $E(y \mid y < \gamma) < \gamma$ for all $\gamma > 0$, we must have $\gamma = 0$ and so $\gamma(0) = 0$. To see that for all $x > 0$, we must have $\gamma(x) > 0$ note that in equation 2 above we have $R(0) < 0$ for all such x. Therefore, by definition of $R(0)$ and γ we must have $\gamma(x) > 0$ whenever $x > 0$ and γ is increasing. ■

With the theorem above, I have fully described the unique equilibrium in the case that the sender discloses for all signal values when she has only one signal. To recap, when she has two signals she discloses the larger. If her larger signal is x then she discloses her smaller signal, y, if it is greater than $\gamma(x)$ where $\gamma : [0,1] \to [0,1]$ is the unique function which satisfies 1 on its entire domain. On the receiver's side, if the sender discloses one signal x, then the expected quality of the sender is given by $\frac{1}{2}(x + \gamma(x))$. If she discloses two signals x and y then the expected quality of the sender is $\frac{1}{2}(x + y)$. Finally, if the sender discloses nothing then her expected quality from the receiver's perspective is 0.

Proving that this is an equilibrium is trivial. Note that a sender will always get a positive payoff from revealing a positive signal. Therefore, it is always correct for the sender with one signal to disclose. It is also correct to disclose at least one signal. Because the sender's payoff is strictly increasing in the revealed signal when she discloses only one, it is then correct to disclose at least the larger signal. Finally, it is optimal to disclose the smaller signal if it is greater than γ evaluated at the larger signal and also to not reveal it if it is less than γ evaluated at the larger signal. Off the equilibrium path, which is only possible if the sender discloses nothing, a single signal that is 0 or a pair of signals both of which are 0, the receiver forms a valid belief and sets the appropriate expected quality for that belief. While I didn't focus on them, the beliefs that lead to the expected quality levels on the equilibrium path correspond to the strategy of the sender and give the correct expected quality. Therefore, this equilibrium satisfies the Perfect-Bayesian equilibrium properties.

2.3 When the One-Signal Sender Never Discloses

Again I will start this section with an example. Suppose that the sender has one signal with probability $p = \frac{3}{4}$ and two with probability $\frac{1}{4}$. Again, assume that if the sender has two signals then they are independent and drawn from the same distribution as if he had one signal. The density is

$$f(x) = \begin{cases} \frac{1}{2} \text{ when } x < \frac{1}{2} \\ \frac{3}{2} \text{ when } x > \frac{1}{2} \end{cases}$$

Consider the following strategies and beliefs. The sender does not disclose if she has a single signal, no matter the value. If she has two signals then the sender discloses neither if their average is less than 0.574 and both if their average is greater than 0.574. If the receiver gets two signals, x and y, he believes that she has these signals with certainty and sets $\hat{q} = 0.574$. If the sender gets no signals then she sets a willingness to pay of 0.574 believing with the appropriate probability that the sender has either a single signal of any value or two signals whose average is less than 0.574. Finally, if the sender discloses a single signal, x, he believes that the sender has two signals x and 0 and guesses $\frac{1}{2}x$.

From the sender's perspective, if she ever discloses a single signal she gets a payoff of at most $\frac{1}{2}$ while she gets a payoff of 0.574 from disclosing nothing. Hence, she will never disclose only one signal. If she has two signals and their average is greater than 0.574 then she can improve by disclosing both signals. Finally, if her signals average less than 0.574, then she would get a smaller payoff from disclosing both. Therefore, she is acting optimally given the strategy of the receiver. The receiver's action is correct given her belief in the off-equilibrium case that the sender discloses a single signal. If the sender discloses two signals then obviously the beliefs correspond to the strategy of the sender and his action is optimal. If she discloses nothing, then her expected quality is given by

$$\begin{aligned} E\left(q \mid \emptyset\right) &= P\left(s=1 \mid \emptyset\right) E_f\left(x\right) + P\left(s=2 \mid \emptyset\right) E\left(\frac{1}{2}\left(x+y\right) \mid \frac{1}{2}\left(x+y\right) < 0.574\right) \\ &= \frac{pE_f\left(x\right) + \left(1-p\right) P\left(\frac{1}{2}\left(x+y\right) < 0.574\right) E\left(\frac{1}{2}\left(x+y\right) \mid \frac{1}{2}\left(x+y\right) < 0.574\right)}{p + \left(1-p\right) P\left(\frac{1}{2}\left(x+y\right) < 0.574\right)} \\ &= \frac{\frac{3}{4}\left(\frac{5}{8}\right) + \frac{1}{4}\left(0.369\right)\left(0.158\right)}{\frac{3}{4} + \frac{1}{4}\left(.369\right)} \\ &= 0.574 \end{aligned}$$

Therefore, the receiver's beliefs correspond to the actions of the sender when on the equilibrium path, the receiver forms some belief off the equilibrium path and the receiver's strategy is optimal given his beliefs.

For any equilibrium of this type, the sender with one signal must not benefit from disclosing it. That is, for all $x \in [0,1]$

$$E\left(q \mid x\right) \leq E\left(q \mid \emptyset\right) \tag{5}$$

If it is off the equilibrium path for the sender to disclose only one signal x, then the receiver's beliefs must be formed with a distribution that puts enough weight on small enough values for 5 to hold. In this sense, the least restrictive belief is the one given in the example - that with probability 1 the sender has two signals the smaller of which is 0. An immediate result of this is that the sender must get a payoff of at least $\frac{1}{2}$ from disclosing nothing.

If the disclosure of a single signal is on the equilibrium path, then 5 must hold with equality. This is the first result.

Theorem 3 *Suppose the sender does not disclose if she has one signal, no matter its value. Then in equilibrium if the sender has two and the average of their values are not equal to* $E\left(q \mid \emptyset\right)$ *she will not disclose exactly one of them.*

Proof. Let Ω be the set of points in $[0,1] \times [0,1]$ such that the sender discloses exactly one of her two signals. Let $E\left(q \mid \emptyset\right)$ be the equilibrium value of the expected quality of the good conditional on her revealing nothing. Let ω be the set of values that z can take such that if the sender has x and z she will reveal only x, and let x^* satisfy $\frac{1}{2}\left(x+x^*\right) = E\left(q \mid \emptyset\right)$. Because the sender does not disclose if she has only one signal, in particular x, optimality implies that $E\left(q \mid \emptyset\right) = E\left(q \mid x\right)$.

Suppose there is a $y \in \omega$ such that $y > x^*$. Then the sender gets a strictly better payoff from disclosing both x and y than from disclosing only x, which contradicts optimality. Therefore, $\omega \subseteq [0, x^*]$. Because the sender will not disclose if she has only x,

$$E(q \mid x) = \frac{1}{2}(x + E(z \mid z \in \omega)). \tag{6}$$

If ω contains points that are smaller than x^* then $E(z \mid z \in \omega) < x^*$ and hence 6 implies that $E(q \mid x) < \frac{1}{2}(x + x^*)$ a contradiction. Therefore, ω contains only x^* and the theorem has been proven. ■

This theorem shows that there may be equilibria where part of the sender's strategy calls for her to sometimes disclose only one of her two signals. However, she will get the same payoff by disclosing one of two signals as from disclosing nothing. In fact, she would get the same payoff from disclosing both. Because this occurs with probability zero it will not affect the equilibrium payoff from disclosing neither. Therefore, once the rest of her strategy has established an expected quality conditional on revealing nothing, she may take any action when she has two signals and their average is equal to this conditional expected quality.

Let's now consider other pairs of signals. Taking $E(q \mid \emptyset)$ as given, if the sender has two signals x and y such that $\frac{1}{2}(x + y) > E(q \mid \emptyset)$ then she is strictly better off disclosing both of her signals than disclosing neither. Because disclosing only one of these signals cannot occur in equilibrium by Theorem 3, she must disclose both in equilibrium. Similarly, if $\frac{1}{2}(x + y) < E(q \mid \emptyset)$ then the sender is strictly better off disclosing neither and she must do so in equilibrium. Graphically, her action when she has two signals is displayed in figure II below. In the red region, the sender discloses nothing. When she has a pair in the green region then she discloses both. Finally, if her signals are on the line between she can take any of the four available actions.

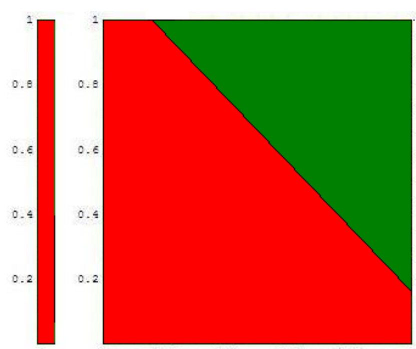


Figure 2: In the red region the sender does not disclose any signals. In the green region she discloses both of her signals.

Thus far I have established the following strategy for the sender. When she has a single signal, she does not reveal it. When she has two signals and their average is below some value k, she discloses neither. If she has two signals and their average is greater than k then she discloses both. If the average of her two signals is equal to k then she does anything.

I have shown that if the sender's strategy is an equilibrium where she discloses nothing if she has a single signal then it will look like the above. Such equilibria may not exist, however. To establish necessary and sufficient conditions on p and f for there to be such an equilibrium, the logical step is to look at the values of k that will lead to an equilibrium.

With a strategy following the above form, her expected quality conditional on having revealed nothing is given by

$$E(q \mid \emptyset) = P(s=1 \mid \emptyset) E_f(x) + P(s=2 \mid \emptyset) E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < k\right).$$

Because

$$\begin{aligned} P(s=1 \mid \emptyset) &= \frac{p}{p+(1-p)P\left(\frac{1}{2}(x+y)<k\right)} \\ \text{and } P(s=2 \mid \emptyset) &= \frac{(1-p)P\left(\frac{1}{2}(x+y)<k\right)}{p+(1-p)P\left(\frac{1}{2}(x+y)<k\right)}, \end{aligned}$$

we have

$$E(q \mid \emptyset) = \frac{pE_f(x) + (1-p)P\left(\frac{1}{2}(x+y)<k\right)E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y)<k\right)}{p+(1-p)P\left(\frac{1}{2}(x+y)<k\right)}.$$

In equilibrium this must equal precisely k. Moreover, if there is a k such that

$$k = \frac{pE_f(x) + (1-p)P\left(\frac{1}{2}(x+y)<k\right)E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y)<k\right)}{p+(1-p)P\left(\frac{1}{2}(x+y)<k\right)} \tag{7}$$

with $k \geq \frac{1}{2}$ there will be an equilibrium of this type. Therefore, most generally the necessary and sufficient conditions are the existence of a $k \geq \frac{1}{2}$ satisfying 7.

Theorem 4 *There is an equilibrium where the sender does not disclose if she has one signal, no matter its value, if and only if there exists a $k \geq \frac{1}{2}$ which satisfies 7.*

Proof. Consider an arbitrary such equilibrium. By Theorem 3 if the sender has two signals whose average is not equal to $\bar{E}(q \mid \emptyset)$ then she will disclose either both or neither, where $\bar{E}(q \mid \emptyset)$ is the expected quality of the sender conditional on the beliefs of the receiver when she discloses nothing. Suppose there is no $k \geq \frac{1}{2}$ which satisfies 7. Optimality implies that if the sender has two signals which average more than $\bar{E}(q \mid \emptyset)$ then she will disclose both and if she has two which average less than $\bar{E}(q \mid \emptyset)$ then she will disclose nothing. Therefore, because the sender is playing this strategy, if there is no k which satisfies 7 then it will be the case that $\bar{E}(q \mid \emptyset) \neq E(q \mid \emptyset)$ which contradicts the assumption of equilibrium.

Let k be greater than or equal to $\frac{1}{2}$ and satisfy 7. Consider the following strategy pair. When the sender has one signal she does not disclose it. If she has two signals and the average of her signals is less than k then she discloses neither. If she has two and their average is greater than or equal to k then she discloses both.

If the sender discloses one signal x then the receiver believes that with probability 1 the sender has two signals and their values are x and 0 and sets $\hat{q} = \frac{1}{2}x$. If the sender discloses two signals x and y then the sender believes that she has two signals x and y with probability one and is willing to pay up to $\frac{1}{2}(x+y)$. If the sender discloses nothing then the receiver believes that the sender has one signal or a signal pair (x, y) such that $\frac{1}{2}(x+y) < k$. He sets $\hat{q} = k$.

If the sender has one signal x then she gets a payoff of k from not disclosing and $\frac{1}{2}x$ from disclosing. Because $k \geq \frac{1}{2}$ it is optimal to not disclose. If the sender has two signals x and y and $\frac{1}{2}(x+y) \geq k$ then she gets at least as high a payoff from disclosing both than neither. Because she gets $\frac{x}{2}$ from disclosing only x and $\frac{y}{2}$ from disclosing only y and $k \geq \frac{1}{2}$ it is optimal to disclose both signals. If she has signals x and y with $\frac{1}{2}(x+y) < k$ then her payoff from disclosing neither is greater than her payoff from disclosing both. Because $k \geq \frac{1}{2}$ she also gets at least as high a payoff from disclosing neither than only one. Therefore, the sender's strategy is optimal given the strategy of the receiver. Because k satisfies 7, the receiver's strategy is in all cases optimal conditional on her beliefs and her beliefs correspond to the strategy of the sender and satisfy Bayes' rule.

Therefore, the above is an equilibrium. Because the sender's strategy calls for her to never disclose when she has a single signal the theorem has been proven. ■

The obvious follow up questions are what conditions ensure that such a k exists and what conditions ensure that there is no such k. I will start with necessary conditions by examining 7. Note that the right-hand side of 7 is a weighted average of $E_f(x)$ and $E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < k\right)$. Because $E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < k\right) < k$, it must be the case that $E_f(x) > k$. By transitivity it is further the case that we must have $E_f(x) > \frac{1}{2}$. This I take as the next result.

Theorem 5 *If $E_f(x) \leq \frac{1}{2}$ then there is no equilibrium such that for all $x \in [0,1]$, $\sigma(x) = \emptyset$.*

A very important corollary of this theorem is that there is no such equilibrium if the distribution of the single signal is symmetric. In particular, this means that there is no equilibrium where the sender never discloses when she has only one signal under the uniform distribution.

With p, f and g given I can go farther. As it turns out, if you write the difference between the right and left hand sides of 7 as a function of k then this function is increasing. This is important because it means that for there to be some $k \geq \frac{1}{2}$ such that 7 holds then the right hand side of 7 must be greater than $\frac{1}{2}$. Combining this with Theorem 4, one can immediately see that if there is an equilibrium where the sender discloses nothing whenever she has one signal then it is unique. Firstly, I will show that the difference between the left and right hand sides of 7 is increasing in k.

Theorem 6 $h(k) = k - \frac{pE_f(x)+(1-p)P\left(\frac{1}{2}(x+y)<k\right)E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k\right)}{p+(1-p)P\left(\frac{1}{2}(x+y)<k\right)}$ *is increasing in k.*

Proof. It is sufficient to show that

$$k\left(p+(1-p)P\left(\frac{1}{2}(x+y)<k\right)\right) - \left(pE_f(x)+(1-p)P\left(\frac{1}{2}(x+y)<k\right)E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k\right)\right) \quad (8)$$

is increasing in k. Let $k > k'$ be arbitrary. It suffices to show that

$$\begin{aligned}
& k\left(p+(1-p)\,P\left(\frac{1}{2}(x+y)<k\right)\right) \\
& \qquad -\left(pE_f(x)+(1-p)\,P\left(\frac{1}{2}(x+y)<k\right)E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k\right)\right) \\
& \qquad\qquad > k'\left(p+(1-p)\,P\left(\frac{1}{2}(x+y)<k'\right)\right) \\
& \qquad\qquad -\left(pE_f(x)+(1-p)\,P\left(\frac{1}{2}(x+y)<k'\right)E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k'\right)\right)
\end{aligned}$$

or

$$p(k-k')+(1-p)\left[\begin{array}{c} P\left(\frac{1}{2}(x+y)<k\right)\left(k-E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k\right)\right)- \\ P\left(\frac{1}{2}(x+y)<k'\right)\left(k'-E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k'\right)\right)\end{array}\right]>0$$

By assumption, $p(k-k')>0$ so it suffices to show that the second term is non-negative. This is true if

$$\begin{aligned}
& P\left(\frac{1}{2}(x+y)<k\right)\left(k-E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k\right)\right) \\
& -P\left(\frac{1}{2}(x+y)<k'\right)\left(k'-E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k'\right)\right)>0
\end{aligned}$$

The first term can be rewritten

$$\begin{aligned}
& P\left(\frac{1}{2}(x+y)<k\right)\left(k-E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k\right)\right) \\
& \qquad = P\left(\frac{1}{2}(x+y)<k\right)E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k\right) \\
& \qquad + \left(P\left(\frac{1}{2}(x+y)<k'\right)+\left(P\left(\frac{1}{2}(x+y)<k\right)-P\left(\frac{1}{2}(x+y)<k'\right)\right)\right)k
\end{aligned}$$

so we have

$$\begin{aligned}
& P\left(\frac{1}{2}(x+y)<k'\right)(k-k')+\left(P\left(\frac{1}{2}(x+y)<k\right)-P\left(\frac{1}{2}(x+y)<k'\right)\right)k \\
& -P\left(\frac{1}{2}(x+y)<k\right)E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k\right) \\
& +P\left(\frac{1}{2}(x+y)<k'\right)E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k'\right)
\end{aligned}$$

Since the first term is positive by assumption, I need only show that

$$\begin{aligned}
&\left(P\left(\frac{1}{2}(x+y)<k\right)-P\left(\frac{1}{2}(x+y)<k'\right)\right)k \\
&-P\left(\frac{1}{2}(x+y)<k\right)E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k\right) \\
&-P\left(\frac{1}{2}(x+y)<k'\right)E_g\left(\frac{1}{2}(x+y)\mid\frac{1}{2}(x+y)<k'\right)
\end{aligned}$$

is non-negative. Rewriting in integral form, we have

$$\begin{aligned}
&\left(\iint_{\frac{1}{2}(x+y)<k} g(x,y)\,dxdy-\iint_{\frac{1}{2}(x+y)<k'} g(x,y)\,dxdy\right)k \\
&-\left(\iint_{\frac{1}{2}(x+y)<k}\frac{1}{2}(x+y)\,g(x,y)\,dxdy-\iint_{\frac{1}{2}(x+y)<k'}\frac{1}{2}(x+y)\,g(x,y)\,dxdy\right) \\
=\;&\iint_{k'<\frac{1}{2}(x+y)<k} k\cdot g(x,y)\,dxdy-\iint_{k'<\frac{1}{2}(x+y)<k}\frac{1}{2}(x+y)\,g(x,y)\,dxdy
\end{aligned}$$

Or

$$\iint_{k'<\frac{1}{2}(x+y)<k}\left(k-\frac{1}{2}(x+y)\right)g(x,y)\,dxdy$$

which is clearly positive and the claim has been proven. ■

If $h(\cdot)$ is increasing then for a $k^* \in \left[\frac{1}{2},1\right]$ that satisfies 7 (in other words a k^* such that $h(k^*)=0$) to exist it must be the case that $h\left(\frac{1}{2}\right)\leq 0$. Moreover, if $h\left(\frac{1}{2}\right)\leq 0$ then there is such a k^*. In other words, that $h\left(\frac{1}{2}\right)\leq 0$ is both a necessary and sufficient condition for the existence of such a k^* and hence for the existence of an equilibrium of this type. Rewriting this gives the next result.

Theorem 7 *There is an equilibrium where for all $x\in[0,1]$ $\sigma(x)=\emptyset$ if and only if f, g and p satisfy*

$$p\left(2E_f(x)-1\right)\geq(1-p)\,P(x+y<1)\left(1-2E_g\left(\frac{1}{2}(x+y)\mid x+y<1\right)\right) \tag{9}$$

then there is an equilibrium where the sender discloses nothing when she has a single signal.

Proof. By the intermediate value theorem and definition of $h(\cdot)$, it suffices to show that when 9 is satisfied then $h\left(\frac{1}{2}\right)\leq 0$. This is true if and only if

$$\begin{aligned}
&\tfrac{1}{2}\left(p+(1-p)P(x+y<1)\right)-\left(pE_f(x)+(1-p)P(x+y<1)E_g(x+y\mid x+y<1)\right) &\leq 0\\
\Leftrightarrow\;&\left(p+(1-p)P(x+y<1)\right)-2\left(pE_f(x)+(1-p)P(x+y<1)E_g(x+y\mid x+y<1)\right) &\leq 0\\
\Leftrightarrow\;&-p\left(2E_f(x)-1\right)+(1-p)P(x+y<1)\left(1-2E_g\left(\tfrac{1}{2}(x+y)\mid x+y<1\right)\right) &\leq 0\\
\Leftrightarrow\;&p\left(2E_f(x)-1\right)\geq(1-p)P(x+y<1)\left(1-2E_g\left(\tfrac{1}{2}(x+y)\mid x+y<1\right)\right) &
\end{aligned}$$

as desired. ■

As I suggested above, that $h(\cdot)$ is increasing gives uniqueness as well.

Theorem 8 *There is at most one equilibrium such that for all $x \in [0,1]$, $\sigma(x) = \emptyset$.*

Proof. By theorem 4 and the definition of $h(\cdot)$, there is such an equilibrium only if k satisfies $h(k) = 0$. Since $h(\cdot)$ is increasing it is also one-to-one. Therefore, there is at most one k such that $h(k) = 0$ and the theorem is proven. ■

2.3.1 An Intermediate Cutoff

I have now covered all equilibria where the sender either always reveals or never reveals when she has only one signal. There also could be equilibria where the sender plays a cutoff strategy with a cutoff that is not at either extreme. Equilibria of this type have very similar properties to the two types given above. For those signal values that are disclosed when the sender has one signal, she behaves in a manner similar to that which is called for when the sender would fully reveal. The same is true for signals that are not revealed by the sender with only one signal.

An example equilibrium strategy for the sender is depicted in Figure #. The sender discloses if she has a single signal that is greater than $\frac{3}{4}$. If she has one signal that is less than $\frac{3}{4}$ then she does not disclose. If she has two signals in the red region then she discloses neither signal. If her signals fall in the green region then she discloses both. Finally, if her signal pair falls in the yellow zone she discloses the larger but not the smaller.

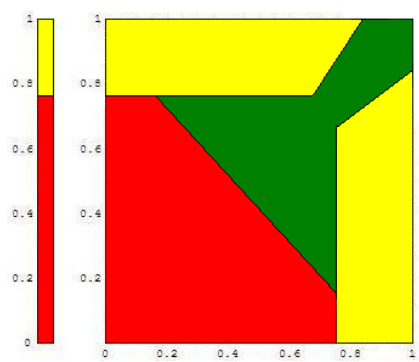


Figure 3: In the red region the sender discloses no signals. In the yellow region she discloses one signal. If she has a pair of signals is in the green region then she discloses both.

The upper boundary of the yellow portion of the graph is the appropriate γ for this probability and distribution. This is, of course, restricted to the $\left[\frac{3}{4}, 1\right]$ interval. The line that splits the green part from the red is the function $2k - x$ so that points on the line have an average of k. This k satisfies an equation similar to 7 that adjusts for the fact that the region goes from 0 to $\frac{3}{4}$ instead of the entire unit interval.

As mentioned above, any equilibrium of this type will have similarities to each of the equilibrium types described in the previous two sections. To examine this, let σ be a strategy such that for some given $c_1 \in (0,1)$,

$$\sigma(x) = \begin{cases} \emptyset \text{ if } x < c_1 \\ x \text{ if } x > c_1 \end{cases}$$

Restricting attention to the $[0, c_1] \times [0, c_1]$ square, if σ is an equilibrium strategy then the action of the sender will be similar to the equilibrium from section the previous section where $\sigma(x) = \emptyset$ for all $x \in [0, 1]$. Firstly, other than possibly on some line segment, the sender will disclose either both signals or neither. Like all results from this section, I will leave out the proofs as they are virtualy identical to a corresponding theorem from the previous section.

Theorem 9 *Let $c_1 \in (0, 1)$ be given and σ be an equilibrium strategy such that $\sigma(x) = \begin{cases} \emptyset \text{ if } x < c_1 \\ x \text{ if } x > c_1 \end{cases}$. In equilibrium, the sender discloses either both of her signals or neither if she has two signals that are smaller than c_1. That is, for all $(x, y) \in [0, c_1) \times [0, c_1)$, $\sigma(x, y) \in \{\emptyset, (x, y)\}$.*

As in the $c_1 = 1$ equilibrium, if the sender has two signals less than c_1 then she will disclose both if their average is greater than some constant $\dot{k}$ and neither if their average is less. The value of this $\dot{k}$ must be equal to the expected quality of the sender if she discloses nothing. That is,

$$\dot{k} = \frac{pF(c_1)\, E_f(x \mid x < c_1)}{pF(c_1) + (1-p)\, P\left(\frac{1}{2}(x+y) < \dot{k}, \max(x,y) < \dot{k}\right)} \quad (10)$$

$$+ \frac{(1-p)\, P\left(\frac{1}{2}(x+y) < \dot{k}, \max(x,y) < c_1\right) E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < \dot{k}, \max(x,y) < c_1\right)}{pF(c_1) + (1-p)\, P\left(\frac{1}{2}(x+y) < \dot{k}, \max(x,y) < c_1\right)} \quad (11)$$

This is analogous to 7 holding for k, with the additional restriction that the larger signal be less than c_1. Again because this is similar to Theorem 4, I will not give the proof.

Theorem 10 *Let $c_1 \in (0, 1)$ be given. If σ is a cutoff-equilibrium strategy then there is a $\dot{k}$ such that 10 holds, for all (x, y) such that $\frac{1}{2}(x + y) < \dot{k}$, $\sigma(x, y) = \emptyset$, and for all (x, y) such that $\frac{1}{2}(x + y) > \dot{k}$, $\sigma(x, y) = (x, y)$.*

Another result which essentially can be taken from the previous section is that there is at most one such $\dot{k}$.

Theorem 11 *There is at most one $\dot{k}$ such that 10 holds.*

I will now move on to signal pairs where the larger signal is greater than c_1. A potential issue is incentive compatibility. It must be the case that for all $x > c_1$, $E(q \mid x) \geq \dot{k}$ or the sender could improve by disclosing nothing if she has one signal that is greater than c_1. I will here take this as given for now and later consider it along with necessary conditions. Let arbitrary $x > c_1$ be given. If the sender has two signals x and y with $x \geq y$ then by assumption the disclosure of only x is at least as good as nothing. That is,

$$E(q \mid x) \geq E(q \mid \emptyset).$$

By the same reasoning as Section 2.2.2, the sender will optimally disclose both x and y as long as y is above some cutoff. Note that since $x > c_1$ and hence $\sigma(x) = x$, there is fundamentally no difference between this situation and that when $c_1 = 0$. That is to say that the cutoff function is simply $\gamma(\cdot)$, defined implicitly by 1. If the smaller signal is greater than γ evaluated at the larger signal then the sender discloses both signals. If it is less then she discloses only the larger signal. Recalling that $\gamma(\cdot)$ is increasing and unique we have

Theorem 12 *Let $c_1 \in (0,1)$ be given. If σ is a cutoff strategy then for every $x > c_1$ there is a unique cutoff $\gamma(x) < x$ such that for all y such that $x \geq y > \gamma(x)$, $\gamma(x,y) = (x,y)$ and for all y such that $x > \gamma(x) > y$, $\sigma(x,y) = x$. The function γ, which maps each point greater than c_1 to its unique cutoff, is increasing.*

Because this γ always exists, there will be an cutoff-strategy equilibrium for c_1 if incentive compatibility is satisfied and if there is a $\dot{k} \geq \frac{1}{2}c_1$ such that 10 holds. Theorem 10 and the definition of incentive compatibility imply that these conditions are also necesary. Recall that in such an equilibrium $\dot{k} = E(q \mid \emptyset)$ and for all $x > c_1$, $\frac{1}{2}(x + \gamma(x)) = E(q \mid x)$. Therefore, for incentive compatibility it must be the case that for all $x > c_1$, $\frac{1}{2}(x+\gamma(x)) \geq \dot{k}$. Because $\gamma(\cdot)$ is increasing this is automatically satisfied if

$$\frac{1}{2}(c_1 + \gamma(c_1)) \geq \dot{k}. \tag{12}$$

Since $\frac{1}{2}(c_1 + \gamma(c_1)) > \frac{1}{2}c_1$, if there is a $\dot{k}$ such that 10 and 12 hold then this $\dot{k} \geq \frac{1}{2}c_1$. Therefore, to show the existence of a cutoff-strategy equilibrium for a given c_1 it is necessary and sufficient to show that there is a $\dot{k}$ such that 10 and 12 both hold.

If the right side of 10 is subtracted from the left and this is written as a function of $\dot{k}$ then this function is increasing.

Theorem 13 *Let $\dot{h}\left(\dot{k}\right) = \dot{k} - \frac{pF(c_1)E_f(x|x<c_1)+(1-p)P\left(\frac{1}{2}(x+y)<\dot{k},\max(x,y)<c_1\right)E_g\left(\frac{1}{2}(x+y)|\frac{1}{2}(x+y)<\dot{k},\max(x,y)<c_1\right)}{pF(c_1)+(1-p)P\left(\frac{1}{2}(x+y)<\dot{k},\max(x,y)<c_1\right)}$. $\dot{h}(\cdot)$ is increasing and $\dot{h}(c_1) > 0$.*

With this result and the above discussion, it is then necesary and sufficient to show that $\dot{h}\left(\frac{1}{2}(c_1 + \gamma(c_1))\right)$ is negative. This can be rewritten:

Theorem 14 *Let $c_1 \in (0,1)$ be given. There is a cutoff-strategy equilibrium with this c_1 if and only if*

$$\begin{aligned} &pF(c_1)\left(2E_f(x \mid x < c_1) - (c_1 + \gamma(c_1))\right) \\ &\geq (1-p)P\left(\frac{1}{2}(x+y) < \frac{1}{2}c_1, \max(x,y) < c_1\right)\left(\begin{array}{c} c_1 + \gamma(c_1) \\ -2E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < \frac{1}{2}c_1, \max(x,y) < c_1\right) \end{array}\right) \end{aligned} \tag{13}$$

Since $E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < \frac{1}{2}c_1, \max(x,y) < c_1\right) < \frac{1}{2}c_1$ the right hand side of 10 is strictly positive and immediate corollary is that $E_f(x \mid x < c_1)$ must be strictly greater than $\frac{1}{2}c_1$. While this doesn't eliminate symmetric distributions for F, it does mean that if the distribution of the single signal is Uniform then the unique equilibrium is for the sender to always disclose a single signal[1].

As before, that $\dot{h}(\cdot)$ is increasing implies uniqueness.

Theorem 15 *Let $c_1 \in (0,1)$ be given. There is at most one cutoff-strategy equilibrium with this c_1.*

2.3.2 Robustness of No-disclosure Equilibria

A major issue with equilibria where no disclosure occurs with positive probability is that they depend heavily on the signal distributions and p. If the sender does not disclose a single signal if it is less than some $c_1 \in (0,1]$ then 10 must hold. The first requirement is that $f(\cdot)$ must be such that $E(x \mid x < c_1) > \frac{1}{2}c_1$. So after a change in f such that this equation no longer holds, there will not be an equilibrium with this cutoff. Another issue is changing p. For any given pair of distributions $f(\cdot)$ and $g(\cdot,\cdot)$ if p is extreme enough in either direction then there is no cutoff equilibrium.

Theorem 16 *For all f and g there is a p' such that for all $p > p'$ there is no equilibrium with $c_1 > 0$. There is also a p'' such that for all $p < p''$ there is no equilibrium with $c_1 > 0$.*

Proof. Let f,g, and c_1 be given. By Theorem 14, 10 must hold in equilibrium. Because F and G are continuous distributions, the right hand side of 10 is continuous. Therefore, it suffices to show that this equation does not hold at either limit. We have

$$\lim_{p\to 0} \frac{\begin{array}{c} pF(c_1)E(x \mid x < c_1) \\ +(1-p)P\left(\frac{1}{2}(x+y) < \dot{k}, \max(x,y) < c_1\right)E\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < \dot{k}, \max(x,y) < c_1\right)\end{array}}{pF(c_1) + (1-p)P\left(\frac{1}{2}(x+y) < \dot{k}, \max(x,y) < \dot{k}\right)}$$
$$= E\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < \dot{k}, \max(x,y) < c_1\right)$$

Because $E\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < \dot{k}, \max(x,y) < c_1\right) < \dot{k}$, 10 does not hold and hence the p'' claim has been proven. Similarly,

$$\lim_{p\to 1} \frac{\begin{array}{c} pF(c_1)E(x \mid x < c_1) \\ +(1-p)P\left(\frac{1}{2}(x+y) < \dot{k}, \max(x,y) < c_1\right)E\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < \dot{k}, \max(x,y) < c_1\right)\end{array}}{pF(c_1) + (1-p)P\left(\frac{1}{2}(x+y) < \dot{k}, \max(x,y) < \dot{k}\right)}$$
$$= E(x \mid x < c_1)$$

If this is equal to $\dot{k}$ then for all $p < 1$, 10 could not hold since the right hand side would be a weighted average of $\dot{k}$ and something larger than $\dot{k}$. So either there is no equilibrium with this c_1, in which case $p' = 0$, or 10 does not hold in the limit. ■

[1] This isn't quite true since the sender could take any action if her quality is zero. No matter her action she gets a payoff of 0 so I ignore this.

In addition to these equilibria only existing for intermediate values of p, another issue is that there is no such equilibrium for certain distributions f and g. For all $c_1 > 0$, 13 must hold. If $f(\cdot)$ is either flat (so that $E(x \mid x < c_1) = \frac{1}{2}c_1$) or is decreasing (so that $E(x \mid x < c_1) < \frac{1}{2}c_1$) then there is no equilibrium where no disclosure is a positive probability outcome. Moreover, if $g(\cdot,\cdot)$ is such that $E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < \frac{1}{2}c_1, \max(x,y) < c_1\right)$ is large for all c_1 then there is no such equilibrium.

3 When the Sender May Have No Signals

Suppose that with probability p_0 the sender has no signals. If she has no signals then her quality is given by q_0, which is assumed to be positive. With probability p_1 the sender has one signal that is drawn from distribution F, which is again assumed to be continuous and have full support on the unit interval. With probability p_2 the sender has two signals that are drawn from distribution G, which is continuous, symmetric and has full support on the unit square. Naturally, it is assumed that $p_0 + p_1 + p_2 = 1$. Assume that the sender and receiver play the same game with an additional strategy assumption that $\sigma(\emptyset) = \emptyset$. I call this the No-Signal Extension Game (NSEG)

Note that in this environment, for all values that the sender discloses if she has one signal she will behave the same way as described in section 2.2.2 when she has two signals. That is, for all x such that $\sigma(x) = x$, the sender will always disclose x if she has two signals x and y such that $y \leq x$. She will also disclose y if it is above a cutoff function evaluated at the larger signal. Similarly, her payoff to revealing one signal is the average of the revealed signal and the cutoff function evaluated at this signal. The only thing that changes in the environment where it is possible that the sender has no signals is that the cutoff function has to be defined in terms of the probabilities. For the remainder of this section let $\gamma(\cdot)$ be implicitly defined by

$$\frac{1}{2}(x + \gamma(x)) = \frac{p_1 x + p_2 G_{y|x}(\gamma)(x + E(y \mid y < \gamma))}{p_1 + 2p_2 G_{y|x}(\gamma)}$$

Note that this is simply restating 1 substituting $p = \frac{p_1}{p_1+p_2}$.

Using this reasoning, it is easy to conclude that in equilibrium the sender will not play the strategy where she reveals for all values if she has one signal.

Theorem 17 *There is no equilibrium of the NSEG such that for all $x \in (0,1]$, $\sigma(x) = x$.*

Proof. If the sender plays such a strategy then by Lemma 1 she will always disclose except possibly if her quality is 0. Because the sender's quality is 0 with probability zero, $E(q \mid \emptyset) = q_0$. Because $\gamma(x) < x$ for all $x \in (0,1]$, $E(q \mid x)$ is bounded above by x. Therefore, for all $x < q_0$ the sender could improve by disclosing nothing if she has one signal equal to x. ■

This theorem means that there are always some signal values such that the sender discloses nothing if she has one signal. To put it in terms of cutoff strategies, for all cutoff-strategy equilibria in the NSEG, $c_1 > 0$. Therefore, all equilibria look like those described in Section 2.3.1. I will not go into full detail on the theorems and proofs as they are again almost identical to those in previous sections. I will only here state necessary and sufficient conditions for the existence of a cutoff-strategy equilibrium for particular values of c_1.

Theorem 18 *Let $c_1 \in (0,1)$ be given. There is a cutoff-strategy equilibrium with this c_1 if and only if there is a $k_0 \geq \frac{1}{2}(c_1 + \gamma(c_1))$ such that*

$$k_0 = \frac{p_0 q_0 + p_1 F(c_1) E_f(x \mid x < c_1)}{p_0 + p_1 F(c_1) + p_2 P_g\left(\frac{1}{2}(x+y) < k_0, \max(x,y) < c_1\right)} \tag{14}$$

$$+\frac{p_2 P_g\left(\frac{1}{2}(x+y) < k_0, \max(x,y) < c_1\right) E_g\left(\frac{1}{2}(x+y) \mid \frac{1}{2}(x+y) < k_0, \max(x,y) < c_1\right)}{p_0 + p_1 F(c_1) + p_2 P_g\left(\frac{1}{2}(x+y) < k_0, \max(x,y) < c_1\right)} \tag{15}$$

As before, the left-hand side minus the right-hand side is an increasing function of k_0. Calling this function $h_0(\cdot)$, for such an equilibrium to exist it must be that $h_0(c_1) > 0$ and $h_0\left(\frac{1}{2}(c_1 + \gamma(c_1))\right)$ be negative. These are also sufficient.

Theorem 19 *Let $c_1 \in (0,1)$ be given. There is a cutoff-strategy equilibrium with this c_1 if and only if $h_0(c_1) \geq 0$ and $h_0\left(\frac{1}{2}(c_1 + \gamma(c_1))\right) \leq 0$.*

Proof. By the intermediate value theorem, $h_0(\cdot)$ is continuous. Therefore, there is a k_0 satisfying the conditions if $h_0(c_1) > 0$ and $h_0\left(\frac{1}{2}(c_1 + \gamma(c_1))\right) < 0$. Therefore by Therem 18 there is such an equilibrium.

Suppose $h_0(c_1) < 0$ ($h_0\left(\frac{1}{2}(c_1 + \gamma(c_1))\right) > 0$). Because $h_0(\cdot)$ is increasing, this implies that there is no $k_0 < c_1$ ($k_0 > \frac{1}{2}c_1 + \gamma(c_1)$) such that 14 holds. By Theorem 18 there is no such equilibrium. ∎

Similarly to the previous section, that h_0 is increasing also implies that there is at most one equilibrium for each c_1.

4 Certification

Suppose in addition to having the option to reveal her signals, the sender is also given the option to certify how many signals she has, perhaps at a small fee. In the business examples this represents the hiring of an outside accounting firm to verify that the sender has disclosed fully. Note that in all of the equilibria given above, there is pooling between the senders with one signal and those with two. Certifying full revelation for a sender with one signal would allow her to separate from a sender with two signals, the larger of which is the same value as hers. Therefore, if a sender with one signal discloses and certifies then her payoff will be the value of the signal. With this certification, I will analyze the set of cutoff-strategy equilibria as I did above when certification was not available. I will first look at the case where certification is costless and then consider positive costs.

In this game, I will represent strategies as a pair (σ, λ) where σ is the function as above which describes the disclosure. It is assumed to satisfy the three properties given in section 2.1 above. Here $\lambda : [0,1] \cup [0,1] \times [0,1] \to \{0,1\}$ determines, for each possible signal and pair of signals, whether or not the sender certifies. If the sender chooses to certify her number of signals and she has one signal (two signals), then the receiver places probability 1 on the sender having one signal (two signals) and probability 0 on her having two signals (one signal).

4.1 Costless Certification

Suppose it is costless for the sender to certify the number of signals that she has. In this case, there will always be full disclosure. The reason for this is that the sender will always certify if she has one signal. This forces the sender with two signals to fully reveal since pooling is impossible. I will now show these separately. In showing that the sender will always reveal if she has two signals I will apply a lemma which states that certification is useless for the sender if she has two signals.

Theorem 20 *If certification is costless then in equilibrium the sender will always disclose and certify if she has one signal that is positive.*

Proof. Suppose the sender discloses nothing on some $\theta \subseteq [0,1] \cup [0,1] \times [0,1]$ such that there is some $x > 0$ such that $x \in \theta$. Then in equilibrium it must be the case that $E(q \mid \emptyset) > 0$. Therefore, the sender will also disclose nothing if she has a single signal with value 0. Hence, there is some x or $(y,z) \in \theta$ such that $x > E(q \mid \emptyset)$ or $\frac{1}{2}(y+z) > E(q \mid \emptyset)$. This contradicts optimality since the sender with such an x could improve by disclosing and certifying and a sender with such a (y,z) could improve by disclosing both. ■

Theorem 21 *If the sender will always disclose and certify if she has a single positive signal then in equilibrium if she has two positive signals then she will disclose both.*

Proof. Suppose the sender's strategy calls for her to disclose and certify if she has a single positive signal. Suppose if the sender's pair of signals is in $\theta \subseteq (0,1] \times (0,1]$ then she will disclose nothing. If θ is non-empty then $E(q \mid \emptyset) > 0$ and hence the sender will also disclose nothing if she has two signals both of which are 0. Therefore, there is some $(x,y) \in \theta$ such that $\frac{1}{2}(x+y) > E(q \mid \emptyset)$ which contradicts optimality. Hence, in equilibrium θ must be empty.

Suppose that if the sender's pair of signals is in $\theta' \subseteq (0,1] \times (0,1]$ then she will disclose only one of her signals. Suppose θ' is not empty and let $(x,y) \in \theta'$. Suppose with no loss of generality that she reveals only x when she has this pair of signals. Because the sender with only x would certify, $E(q \mid x) = \frac{1}{2}(x + E(z \mid \sigma(x,z) = x))$. Since $(x,y) \in \theta'$ and $y > 0$ this must be strictly positive. Therefore, it is optimal for the sender to disclose only x if she has $(x,0)$. Therefore, there is some w such that $\sigma(x,w) = w$ and $\frac{1}{2}(x+w) > E(q \mid x)$, which contradicts optimality. Therefore, in equilibrium θ' must be empty.

Because θ and θ' are both empty in equilibrium, the sender will fully disclose if she has two positive signals. ■

A corollary of the above two results is that if certification is costless then the sender will disclose any positive signals that she has. Similarly to the result from section 2.2.2 where the sender always discloses one signal, if the receiver gets no signals then she must believe that the sender has quality 0. Also, if the receiver gets one signal but no certification she must believe that the sender has two signals and the one not disclosed is 0. The payoff to the sender will always be her true quality.

A situation where the cost is zero would occur if the sender were allowed to state that how many signals she had as well as their values and this was credible, for example if the cost of lying were large. The above results mean that in such games there will always be full disclosure.

4.2 Costly Certification

Suppose it costs some $\varepsilon > 0$ for the sender to certify the number of signals that she has. Unlike under costless certification, there may be multiple equilibria. The set of cutoff-strategy equilibria looks very similar to those given in section 2.2 when certification was not an option. As in that section, I will first analyze the game when the sender discloses if she has a single positive signal (so $c_1 = 0$) and then consider possible equilibria where $c_1 \in (0, 1]$.

4.2.1 Full One-Signal Disclosure

As before, if the sender will always disclose if she has one positive signal then she will also always disclose at least one signal. Certification doesn't affect this. I'll here restate lemma 1.

Lemma 2 *If σ is an equilibrium pure strategy such that $\sigma(x) = x$ for all $x > 0$ then $E(q \mid 0) = E(q \mid \emptyset) = 0$.*

That the sender will disclose at least one signal if she has two, at least one of which is positive, is an immediate result of this.

I will now consider the affect of certification on the actions of the sender when she has two signals. Let $\Lambda = \{x \in [0, 1] \mid \lambda(x) = 0\}$, the set of values at which the sender would not certify if she has one signal. If the sender has two signals and the larger is in Λ then she will take the same action as before. That is, she will reveal the larger signal and will also reveal the smaller signal if it is larger than γ evaluated at the larger signal. If the larger the two signals is in Λ^c then the sender will reveal both signals. Intuitively, on the set of values where the sender would certify the sender with two signals is not able to pool and must fully disclose. On the set of values where the the sender does not certify, nothing changes. I show this in the next theorem.

Theorem 22 *In equilibrium, for all x and y such that $x \geq y > 0$ if $x \in \Lambda^c$ then $\sigma(x, y) = (x, y)$, if $x \in \Lambda$ then $\sigma(x, y) = x$ if $y < \gamma(x)$ and $\sigma(x, y) = (x, y)$ if $y > \gamma(x)$.*

Proof. Suppose the sender has two signals x and y such that $x \geq y > 0$ with $x \in \Lambda$. By Lemma 1, the sender is strictly better of disclosing x than nothing. By the definition of γ, she is better off disclosing only x than both x and y when $y > \gamma(x)$ and worse off if $y < \gamma(x)$. If $y \in \Lambda$ then γ being increasing implies that she is strictly better off disclosing only x than disclosing both x and y. Therefore she will take the action stated whenever $x \in \Lambda$.

Suppose the sender has two signals x and y such that $x \geq y > 0$ with $x \in \Lambda^c$. Then if she discloses only x then the receiver will believe that with probability one she has two signals. By unraveling she is strictly better off disclosing both x and y. The same holds if she discloses only y if $y \in \Lambda^c$. If $y \in \Lambda$ then she will get a payoff of $\frac{1}{2}(y + \gamma(y))$ from disclosing only y. Since $x > y > \gamma(y)$ she is strictly better off disclosing both x and y. Therefore, if $x \in \Lambda^c$ the sender will disclose both signals and the theorem has been proven. ■

I will now analyze the set of values over which the sender will certify if she has one signal. Optimal certification with full disclosure can then be combined with Theorem 22 to form an equilibrium. The sender will certify if the benefit to her from doing so outweighs the cost of certification. If she discloses and certifies then her payoff is the value of her signal minus the cost of certification. If she does not certify then her payoff is the average of her signal and γ evaluated at her signal. Therefore, the sender will certify for all x such that

$$x - \varepsilon > \frac{1}{2}\left(x + \gamma\left(x\right)\right)$$

and will not certify for all x such that

$$x - \varepsilon < \frac{1}{2}\left(x + \gamma\left(x\right)\right).$$

Again, if the left hand side equals the right hand side then the sender is indifferent and can take either action or mix.

As it turns out, there is always a single cutoff. Above this cutoff the sender will always certify and below it she will not. This result holds because $\gamma\left(\cdot\right)$ is concave.

Theorem 23 *If for all $x \in (0,1]$, $\sigma\left(x\right) = x$ then there is a unique $x^* \in (0,1]$ such that for all $x < x^*$ the sender will not certify and for all $x > x^*$ she will certify.*

Proof. By Theorem if $\lambda\left(x\right) = 1$ then if the sender has two signals x and some y such that $x \geq y > 0$ she will disclose x and if $y > \gamma\left(x\right)$ then she will also disclose y. The payoff to the sender if she has a single signal with value x is and discloses without certifying is then $\frac{1}{2}\left(x + \gamma\left(x\right)\right)$. If she certifies and discloses her payoff is $x - \varepsilon$. Therefore, it is optimal to disclose and certify if

$$x - \varepsilon > \frac{1}{2}\left(x + \gamma\left(x\right)\right).$$

This can be rewritten

$$x - \gamma\left(x\right) > 2\varepsilon.$$

Since 2ε is fixed the claim will be proven if it can be shown that $x - \gamma\left(x\right)$ is increasing in x.

Let arbitrary x and x' such that $x > x'$ be given. If $x - \gamma\left(x\right) - \left(x' - \gamma\left(x'\right)\right) > 0$ then it will be shown that $x - \gamma\left(x\right)$ is increasing in x. Recall that by definition of γ, we have

$$\frac{1}{2}\left(x + \gamma\right) = \frac{px + \left(1-p\right)G_{y|x}\left(\gamma\right)\left(x + E\left(y \mid y < \gamma\right)\right)}{p + 2\left(1-p\right)G_{y|x}\left(\gamma\right)}.$$

This can be rewritten

$$x - \gamma\left(x\right) = 2\left(\frac{1-p}{p}\right)G_{y|x}\left(\gamma\right)\left(\gamma - E\left(y \mid y < \gamma\left(x\right)\right)\right).$$

Let $\gamma = \gamma\left(x\right)$ and $\gamma' = \gamma\left(x'\right)$ respectively. We have

$$x - \gamma - \left(x' - \gamma'\right) = 2\left(\frac{1-p}{p}\right)\left(G_{y|x}\left(\gamma\right)\left(\gamma - E\left(y \mid y < \gamma\right)\right) - G_{y|x}\left(\gamma'\right)\left(\gamma' - E\left(y \mid y < \gamma'\right)\right)\right).$$

This is greater than 0 if and only if

$$\begin{aligned} G_{y|x}(\gamma)(\gamma - E(y \mid y < \gamma)) - G_{y|x}(\gamma')(\gamma' - E(y \mid y < \gamma')) &> 0 \\ \Leftrightarrow \left(G_{y|x}(\gamma)\gamma - G_{y|x}(\gamma')\gamma'\right) - \left(G_{y|x}(\gamma) E(y \mid y < \gamma) - G_{y|x}(\gamma') E(y \mid y < \gamma')\right) &> 0 \end{aligned}$$

Rewriting in integral form we have

$$\begin{aligned} \left(\int_0^{\gamma} \gamma g_{y|x}(y)\, dy - \int_0^{\gamma'} \gamma' g_{y|x}(y)\, dy\right) - \left(\int_0^{\gamma} y g_{y|x}(y)\, dy - \int_0^{\gamma'} y g_{y|x}(y)\, dy\right) &> 0 \\ \Leftrightarrow \int_0^{\gamma} (\gamma - \gamma') g_{y|x}(y)\, dy + \int_{\gamma'}^{\gamma} \gamma g_{y|x}(y)\, dy - \int_{\gamma'}^{\gamma} y g_{y|x}(y)\, dy &> 0 \\ \Leftrightarrow \int_0^{\gamma} (\gamma - \gamma') g_{y|x}(y)\, dy + \int_{\gamma'}^{\gamma} (\gamma - y) g_{y|x}(y)\, dy &> 0 \end{aligned}$$

The last line clearly holds since $\gamma > \gamma'$. Therefore, $x - \gamma(x)$ is strictly increasing in x and the theorem has been proven. ■

An example of this equilibrium is shown below. In Figure 4 one can see the process for determining the x^*. The horizontal axis represents the value of the signal. The black line is the 45-degree line. The y value of any given point on this line is the true quality of the sender with such a signal. The green line is $\gamma(x)$. Without certification the payoff to the sender if she discloses x is the average of x and $\gamma(x)$. Therefore, the payoff function for a given signal value is the average of the black and green lines. The blue line depicts this. Finally, the red line represents the payoff to the sender if she discloses a given signal and certifies. It is simply the 45-degree line shifted down by the cost of certification. Note that the red and blue lines cross at $x = 0.42$. Above this value, the red line is above the blue line and below the opposite is true. From this we can see that the sender will find it optimal to certify if she has one signal that is above 0.42 and will not certify if she has a signal that is smaller. Figure 5 depicts the equilibrium. In the yellow region the sender discloses one signal and does not certify. In the brown region she discloses and certifies. Finally, in the green region she discloses two signals.

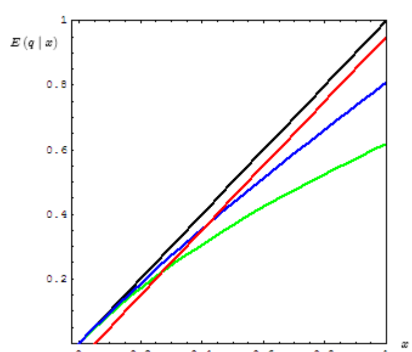


Figure 4: The black line represents the actual quality of a sender who has a single signal with value x. The red line is this shifted down by some ε - giving the payoff to certifying and disclosing. The blue line is the payoff to the sender of disclosing x without certifying - the average of $\gamma(x)$, which is the green line, and the $45°$ line.

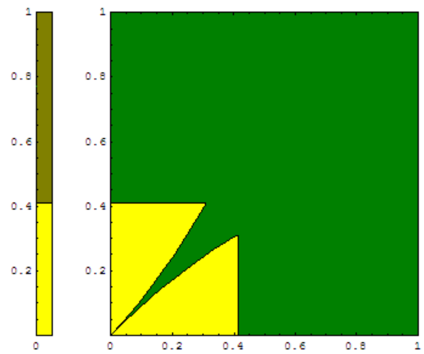


Figure 5: In the green region the sender discloses both signals and does not certify. In the yellow region she discloses one signal and does not certify. If her signal is in the brown region then she discloses it and also certifies.

4.2.2 Partial One-Signal Disclosure

Suppose σ satisfies the cutoff-strategy requirements with $c_1 > 0$. Then in equilibrium it must be the case that for all $x < c_1$ the sender is at least as well off disclosing nothing than disclosing x and certifying if x is her only signal. In addition, for all signal values above c_1 the sender behaves as in the equilibrium with full one-signal disclosure and certification. That is, there is some $x^* \in [c_1, 1]$ such that the sender discloses and certifies if she has one signal above x^* and discloses without certification if her signal is between c_1 and x^*. If the cost is low enough then there could be certification over all disclosed values. Similarly, if the cost is high enough it may never be used. If the sender has two signals and the larger is above this x^* then she will disclose both. Otherwise she will take whatever action she would have in the respective equilibrium without certification. The reasoning is the same as the full one-signal disclosure equilibrium with certification given in the previous section.

In summary,

Theorem 24 *Suppose* $c_1 \in (0,1)$ *and* $\sigma(x) = \begin{cases} \emptyset & x < c_1 \\ x & x > c_1 \end{cases}$. *In an* ε*-certification equilibrium the following must hold:*

1. $c_1 - \varepsilon \leq E(q \mid \emptyset)$
2. *There exists an* $x^* \in [c_1, 1]$ *such that* $\lambda(x) = \begin{cases} 0 & x < x^* \\ 1 & x > x^* \end{cases}$.
3. *For this* x^*, $\sigma(x,y) = (x,y)$ *whenever* $\max(x,y) > x^*$.
4. *If* $\max(x,y) \in (c_1, x^*)$ *then* $\sigma(x,y) = \begin{cases} \max(x,y) & \min(x,y) < \gamma(\max(x,y)) \\ (x,y) & \min(x,y) > \gamma(\max(x,y)) \end{cases}$.
5. *There exists a* $\dot{k}$ *such that if* $\max(x,y) < c_1$ *then* $\sigma(x,y) = \begin{cases} \emptyset & \frac{1}{2}(x+y) < \dot{k} \\ (x,y) & \frac{1}{2}(x+y) > \dot{k} \end{cases}$ *where* $\dot{k}$ *satisfies 10.*

Proof. 1 must hold due to optimality. Proving 2, 3 and 4 are similar to the equivalent statements in theorems 23 and 22. 5 is easily seen since the equilibrium calls for non-disclosure if the sender has a signal less than c_1 and the sender will not certify for any of these signal values. ■

Of particular importance is claim 1. A result of it is that given a particular positive c_1, if ε is small enough then there is no ε-certification equilibrium where the sender discloses if she has one signal greater than c_1 and does not if her signal is smaller. Contrast this with the full disclosure case where $c_1 = 0$, for which there is always an equilibrium. This raises another robustness issue with equilibria where $c_1 > 0$. They are not robust to ε certification for very sufficiently small ε.

In figure 6 one can see different equilibria for the same c_1. The first panel is the equilibrium for the case where the cost of certification is sufficiently high that the option is never used. This is exactly the same as Figure #. The second panel shows the equilibrium where ε is such that if the sender has one signal then she will disclose and certify for large values, neither disclose nor certify for small values and disclose but not certify for values in between. The third panel shows the case where certification costs are small enough that the sender will always certify if she discloses a single signal.

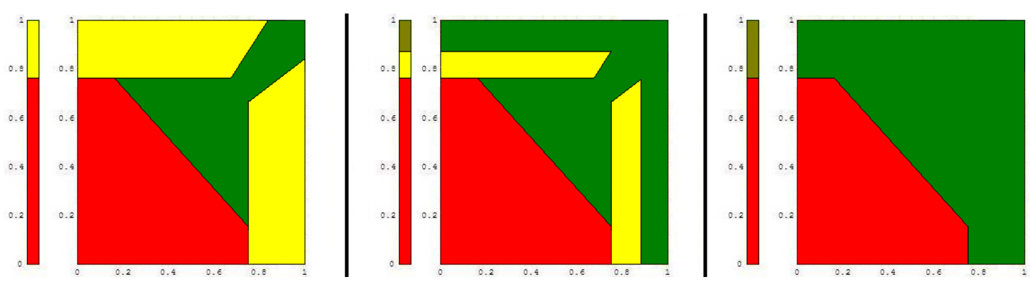


Figure 6: Each frame depicts an equilibrium for different costs of certification. In the first frame certification is too costly to use. In the third frame the costs are sufficiently low that it is always used when one signal is disclosed. The second frame shows an in-between case. In all frames she discloses and certifies in the brown region, discloses one signal but does not certify in the yellow region, discloses both but does not certify in the green region and discloses nothing and does not certify in the red region.

Note that as in the ε-certification equilibrium with $c_1 = 0$, there is always more disclosure so long as ε is small enough that some certification occurs. What cannot be seen is that if ε is small enough so that some, but not full, certification occurs then further decreasing the cost of certification will lead to even more disclosure. If there is not full certification at the given ε, as in the second panel, then the certification range will expand which in turn will cause the full-disclosure range to expand.

5 Multiple Senders with One Receiver

Suppose you have a single receiver and multiple senders. This situation may arise when there are multiple senders of similar products. The senders reveal information about their products to the receiver, who then chooses from which sender to make a purchase. Due to random variation, the senders could have different amounts of information about their products. For example, suppose there is an investor who is interested in buying stock. Some firms in the industry will know their sales information for the previous month while others may not. This information difference could be the result of the structure of the firms, their industries, or something else. The existence of this information by itself may or may not be an indication of the quality of the firm.

A situation that is perhaps more directly important to Economists is academic letters of recommendation. When there are academic openings, whether they be for graduate school or academic jobs, there are applicants from a variety of situations. Some come from large departments, others from very small departments. While the standard number of letters is 3, having letters from faculty members with whom one has worked closely is significantly better than from those from outside the department. Those coming from larger departments have more people available in a particular field from which they may get more informative letters than those from smaller schools.

The design is very similar to the one-sender one-receiver model from Section 2.1 with the obvious addition of more senders. More formally, suppose there are N senders and one receiver. Each sender independently has one signal with probability p and two with probability $1-p$. Senders with only one signal draw it from distribution F with full support on $[0,1]$. Senders with two signals draw them from joint density $g(\cdot,\cdot)$. The quality of sender i is the average of her signal(s). The senders know the above and their own signal(s), but do not know the quantity or value of the signals of any other sender. The senders simultaneously send any subset of their signals to the receiver. The receiver then ranks the senders in order by expected quality.

Because of the information structure of the game, it is equivalent to the original game for each sender-receiver pair. Each sender will reveal signals to maximize her expected quality. Therefore, an equilibrium to this game will be each pair (every sender combined with the receiver) playing strategies that are described by an equilibrium given above. To simplify the analysis, I will always assume that each pair is playing the equilibrium where the sender discloses fully if she has one signal. As this is the only equilibrium that exists for all values of f and p, this isn't very restrictive.

Efficiency and fairness become concerns in the environment with multiple senders. This is due to the pooling of some of the two-signal senders. Senders with two signals could appear to be of higher quality than a sender with one signal even if they are in fact lower. The reverse does not hold. This asymmetry lessens efficiency because this leads to unfairness. A perfectly efficient outcome in this setting would be one where the receiver's ranking is exactly how they would be ranked by actual quality. This is similar to the notion of efficiency in private value auctions where an auction is efficient if with probability one the winner of the auction is the bidder who values the good the most.

To assess efficiency and fairness, I will consider three different disclosure rules. The first is just the standard game. Senders may reveal nothing, one signal, or two if they have them. The second disclosure rule is that senders may disclose only one signal. If a sender discloses two signals then she will automatically be placed at the bottom of the ranking. In order to avoid potential issues with incentive compatibility, I'm assuming that either an outside agent sets this rule and enforces it or that there is some commitment device that prevents the receiver from considering a sender who discloses two large signals. The third rule is full disclosure. This could come about if full disclosure is mandatory or if, as was seen in the previous section, the senders are allowed to verify the number of signals that they have at no cost. Finally, I will also consider the possibility that the receiver ignores all information from the senders and ranks them randomly.

For notation, I use s_i and s_j to denote the two senders. Unless otherwise noted, this will be two random senders who have not yet taken part in the first lottery to determine how many signals that they have. At times I will compare a random sender with one signal to a random sender with two signals, two random senders with two signals, and so on. I may also assign them signal values.

In general let $s_i \succ s_j$ indicate that sender i is of higher expected quality than sender j after they have revealed their signals and hence is ranked higher. I call the standard game with all strategies available to the players the unrestricted disclosure rule and attach the subscript U to the relation, $\succ_U$. The restricted game where players may disclose at most one signal I call the one-signal only rule and use this subscript O, $\succ_O$. The relation under the full-disclosure rule is denoted $\succ_F$. Finally, let q_i be the actual quality of sender s_i. Using this notation, I will now more carefully define fairness and efficiency. I define these terms using pairwise comparisons.

Definition 25 *Disclosure rule α is efficient if $P(s_i \succ_\alpha s_j \mid q_i > q_j) = 1$. Disclosure rule α is more efficient than rule β if $P(s_i \succ_\alpha s_j \wedge q_i < q_j) + P(s_j \succ_\alpha s_i \wedge q_j < q_i) < P(s_i \succ_\beta s_j \wedge q_i < q_j) + P(s_j \succ_\beta s_i \wedge q_j < q_i)$*

Definition 26 *Disclosure rule α is fair if $P(s_i \succ_\alpha s_j) = P(q_i > q_j)$. Disclosure rule α is more fair than rule β if $|P(s_i \succ_\beta s_j) - P(q_i > q_j)| < |P(s_i \succ_\beta s_j) - P(q_i > q_j)|$.*

The way I have defined efficiency, a disclosure rule is inefficient if with any probability a sender of lesser quality is chosen over a sender of higher quality. Given that pooling occurs in any equilibrium with less than full disclosure, the only possible way to get this would be full disclosure. Fairness is more plausible as it allows s_i, for example, over some possible signal values to be chosen despite being lower in actual quality if this is offset by her not being chosen for other values despite being higher in actual quality.

If one of the three disclosure rules above is in place, then there are only two relevant types of pairings - those where one sender has one signal and the other has two and those where both senders have two signals. Increasing fairness will generally be done by making the one-signal sender better off in the first type. This will, however, be costly in efficiency terms for those pairings of the second type. Whether the increase in efficiency from the one-signal sender and two-signal sender pairings offset that depends on the distributions and p.

In giving the result for fairness I rely on two lemmas. The first is that when analyzing fairness I only need to consider pairs where one sender has one signal and the other sender has two signals. This is an immediate result of the fact that two senders with the same number of draws take their draws independently from the same distribution.

Lemma 3 *The only pairing that is affected by fairness is the one-signal-two-signal.*

To simplify notation, I will denote the two senders as s_1 and s_2 where the subscript gives the number of signals that the given sender has. The second lemma states that the sender with one signal will never undeservingly be ranked above the sender with two signals in either the standard game or under the restriction that the senders may only send one signal.

Lemma 4 *There is no set of signals such that $q_1 < q_2$ but $s_1 \succ_A s_2$. The same holds for $\succ_R$*

Proof. Let x and (y, z) be the signal holdings of senders 1 and 2 respectively. Let x, y, and z be given and arbitrary such that $x < \frac{1}{2}(y + z)$. Under the standard game, the expected quality of sender 1 is $\frac{1}{2}(x + \gamma(x)) < x < \frac{1}{2}(y + z)$. Since the expected quality of sender 2 is at least $\frac{1}{2}(y + z)$, sender 1 will be ranked below sender 2. If senders may send only one signal then sender 1 will send x and 2 will send $\max(y, z)$. Since $\max(y, z) > \frac{1}{2}(y + z) > x$, sender 2 will send a strictly higher signal and hence be ranked higher. ■

I am now ready to give the principal result on fairness.

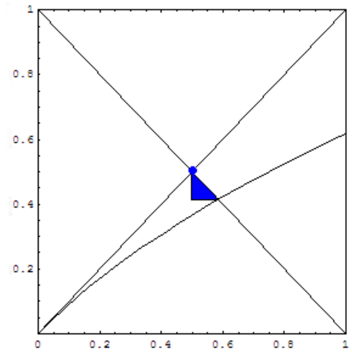
Theorem 27 *The full disclosure rule is fair. It is more fair than the one-signal only disclosure rule, which is more fair than the unrestricted disclosure rule.*

Proof. Under full disclosure the receiver knows the true quality level of each sender. Therefore, he will rank them appropriately by true quality and hence $P(s_1 \succ_F s_2) = P(q_1 > q_2)$ so the rule is fair.

To show that the one signal only disclosure rule is unfair and hence less fair than the full disclosure rule it suffices to show that with positive probability senders 1 and 2 draw signals such that sender 1 has higher actual quality then sender 2 but from the perspective of the receiver sender 2 appears to have higher expected quality. Consider the following subset of $[0, 1] \times [0, 1] \times [0, 1]$. Let x range from 0 to 1. For each value of x let y range from x to $\frac{3}{2}x$ and z range from 0 to $\frac{1}{2}x$. In this set, let x denote the signal of sender 1 and y and z the signals of sender 2. Over this entire set, $y > x$ and hence sender 2 will send a higher signal to the receiver. However, it is also the case that $x > \frac{1}{2}(y + z)$ so the sender with two signals wins unfairly. By Lemma #, sender 1 never wins unfairly and hence the rule is unfair.

To see that the one-signal rule is more fair than the unrestricted rule, note that the set described in the paragraph above is the only set where sender 2 unfairly wins over sender 1. I will show that the one-signal rule is less fair by showing that there is another set of three signals that occurs with strictly positive probability in which sender 2 unfairly wins over sender 1. Let x range from 0 to 1. For every x, let y and z fall in the area bounded above and to the right by the line that passes through x perpendicular to the 45-degree line, to the left by the vertical line passing through x and below by the horizontal line which passes through the larger of $\frac{1}{2}x$ and the intersection of the reverse 45-degree line and γ. An example for $x = \frac{1}{2}$ is depicted in Figure #. This set has strictly positive measure. Since f and g are assumed to have full support it must occur with positive probability. Every point in the set is a triple where if sender 1 has signal x, and sender 2 has signals y and z then sender 1 will be of higher quality but will appear to be lower in expected quality once they have revealed their signals. The set does not overlap with that described in the previous paragraph and therefore sender 2 unfairly wins with strictly higher probability. Because sender 1 never unfairly wins the unrestricted disclosure rule is strictly less fair. ■

The first lemma states that a sender with one signal never has an (ex-post) unfair advantage over a sender with two signals when the senders may disclose whatever they wish or



when they may disclose at most one signal. The theorem simply states the restricted rule is more fair to the one-signal sender. Combining these results, if we restrict our attention to pairings of this type then any increase in fairness is also an increase in efficiency.

Unfortunately, it does not guarantee an increase in efficiency overall. For this we need that the probability of having only one signal be high enough. This value depends on the distributions.

Theorem 28 *For any given f and g, there is a $\bar{p}$ such that if $p > \bar{p}$ then the unrestricted rule is more efficient than the one-signal rule and if $p < \bar{p}$ then the reverse holds.*

Proof. Because the sender always ranks correctly a pairing of two senders with one signal under any rule, efficiency is given by 1 minus the probability of a one-signal two-signal sender pair times the probability that the two-signal sender unfairly wins minus the probability that both senders in a pair have two signals times twice the probability that one of them unfairly is ranked higher. Because there is more pooling, the probability that a sender with two signals is unfairly ranked above another sender with two signals is greater under the one-signal rule than the unrestricted. As shown above, the opposite is true for pairings where one sender has one signal and the other has two. Let $\phi(\alpha)$ be the probability that a two-signal sender is unfairly ranked above when paired with another two-signal sender under rule α and $\psi(\alpha)$ be the probability that a two-signal sender unfairly is ranked higher when paired with a one-signal sender. The probability that two two-signal senders are paired is $(1-p)^2$ while the probability of a one-signal two-signal pairing is $2p(1-p)$. The efficiency of rule α can be written

$$1 - 2p(1-p)\psi(\alpha) - 2(1-p)^2\phi(\alpha)$$

The unrestricted rule is more efficient if

$$\begin{aligned} 1 - 2p(1-p)\psi(U) - 2(1-p)^2\phi(U) &> 1 - 2p(1-p)\psi(O) - 2(1-p)^2\phi(O) \\ \Leftrightarrow \left(\frac{1-p}{p}\right)(\phi(O) - \phi(U)) &> \psi(U) - \psi(O) \end{aligned}$$

Because $\frac{1-p}{p}$ is strictly decreasing in p and $\phi(O) - \phi(U)$ and $\psi(U) - \psi(O)$ are positive constants the claim has been proven. ■

These two theorems have major policy implications. If the receiver is interested primarily in fairness then it is better to restrict the strategy space so that senders with multiple signals can send only as many as those senders with the minimum number of signals. For example, if an employer wishes to place special emphasis on the fairness of the hiring process they would do well to limit the number of letters of recommendation to an amount that everyone could produce. There would still be an advantage for those people who are better connected, but it would be attenuated.

On the other hand, if the receiver cares mainly about the probability of choosing the most qualified candidate, then depending on the distribution of people with more or fewer signals and the value of those signals, he may want to limit the maximum number of signals to be sent or may not. In any case, full disclosure will lead to a perfectly fair and efficient ranking.

6 Conclusion and Further Research

I have developed a model that captures the important aspects of a variety of situations in order to answer questions that have not been the focus of the literature on disclosure. Notably, I find that unraveling does not hold and there is always less than full disclosure if the receiver is uncertain of how much information the sender has. This has important policy implications because it suggests that mandatory disclosure laws may be necessary if the government wishes for full disclosure. Surprisingly, I have also found that there may be equilibria where with positive probability nothing is disclosed, even if it is common knowledge that the sender has a positive number of signals. While these equilibria are not robust to changes in parameters, they indicate that there could be even less disclosure than most would think.

I further extended the model to analyze the effect of allowing the sender to certify, at a small cost, how many signals she has. If the cost is small enough so that the sender sometimes wishes to certify, then there is always more disclosure than if the sender is not able to do so. When this cost goes to zero there is full disclosure. This suggests another two ways for the government to cause the full disclosure outcome - allow for the informed agents to state how much information they have and make the penalty for falsely reporting very high. Another way would be to subsidize independent accounting firms that would make certification free for the informed agents, or for the government to perform whatever necessary audits and certify themselves.

Finally, I analyze the game with multiple senders and one receiver. Among other places, this situation arises in practice in applications for academic jobs or graduate school. The primary questions there are fairness and efficiency. I find that allowing the senders to submit as much information as they wish is neither fair nor efficient. This is so because senders with more information are at an advantage over senders with less information. The receiver can improve fairness by limiting the number of signals that the senders may transmit. This provides a theoretical justification for the common practice of allowing exactly three letters of recommendation. However, this may lead to a less efficient outcome in the sense that it may make it more likely that the ex-post ranking of senders by the receiver is less likely to be accurate. The reason for this is that restricting information makes ranking senders with a lot of information more difficult than just letting them submit as many signals as they wish.

One important assumption in this paper is that the sender seeks to maximize her current expected quality in the eyes of the receiver. Getting back to the firm manager example, this is equivalent to the manager wanting to maximize the current capitalization. While CEOs have large incentives to maximize current capitalization and are often under short contracts, they will also take the future into consideration. This could significantly change the results. Firstly, the manager of the firm may not want investors to overvalue the firm today because that could cause them to undervalue it tomorrow. Or, as in Crawford and Sobel [2], perhaps the manager today wants investors to overvalue the firm, but not by too much. The public's reaction in the future is especially important if people are averse to being deceived and feel that withholding information is deceptive. The manager may wish to establish a commitment device so that in future periods there is full disclosure, even of bad information.

I take the number of signals that the sender has as exogenously determined. In some examples this may not be completely reasonable. For example, a firm could shift its research and development or accounting to make more or fewer reports available to management, but this would be a long-run concern. Because I am focusing on the short-run and looking at what the firm will do given the information it has at the current time the exogeneity assumption is reasonable. Getting relevant information takes time, so it is safe to assume that the firm does not have control over the quantity of information immediately available. However, in some industries information may be available virtually instantaneously. In these environments, there is a cost to attain this information and it is not always available when a manager seeks it. If this is the case then even if it is endogenously attained, in equilibrium the receiver will not know how much information the sender has. Therefore, there will be less than full disclosure for the same reasons I've given in this paper. When considering an endogenous mechanism for the amount of information that a sender has, timing is important. In some situations a firm may try to attain a certain number of reports and the manager will have as many as were possible. In others, a firm may get a report and then based on its findings decide whether or not to seek out more information. This allows strategies where a firm that has some unfavorable information increases effort to get more information while a firm with only favorable information does not.

Another assumption in this paper is that there be one receiver with a quadratic-loss utility function. This is equivalent to the receiver having linear utility in quality and the sender setting a price after the disclosure stage. In real-world environments there tend to be several consumers with different utility functions. Some may be very sensitive to quality and others may not. Similarly, some may have different levels of risk aversion. If the sender reveals her signal(s) in a public manner and cannot distinguish the receivers from one another then this will affect both the disclosure decision and the price that she sets. It also raises the issue of efficiency, which did not come up in my principal model. The efficiency issue arises because the senders with little information will not be able to sell to some risk-averse agents to whom they could sell if their true quality was known. Forcing full disclosure or allowing for certification could lead to more mutually beneficial transactions. A natural extension of this is a more complete environment with several receivers and a few senders. This would likely change the outcome significantly due to price-competition.

Another potential extension is changing the definition of quality. Here, the quality of the sender is the average of the signal(s) that she knows. There are some similar situations where this is not realistic. For example, if a firm is selling its product in several countries and has sales figures for some, but not all, of them then the value of the firm depends on sales in all countries, not just those it knows about. I considered this case but have not included it in this paper because unlike in this paper the correlation of signals is particularly important. I am currently working on a paper to analyze this more carefully.

A different research direction is to consider the plausibility of the rationality assumption. Many proponents of mandatory-disclosure legislation push for it to prevent firms from hiding information in order to benefit from fooling investors. When rationality is assumed, this is not possible because the investor will base his actions on the expected quality of the firm. Therefore, if such protection is necessary then investors and consumers must not be properly updating their expectations. Anecdotal evidence also suggests that people seem to have trouble dealing with partial data sets that have been selected by a biased agent. If agents are irrational, then there are major implications on the issue of mandatory disclosure. I am currently working on three projects that test for rationality and characterize any bias receivers may have in these games.

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