

INCOMPRESSIBLE SURFACES, HYPERBOLIC VOLUME, HEEGAARD GENUS AND HOMOLOGY

MARC CULLER, JASON DEBLOIS, AND PETER B. SHALEN

ABSTRACT. We show that if M is a complete, finite-volume, hyperbolic 3-manifold having exactly one cusp, and if $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 6$, then M has volume greater than 5.06. We also show that if M is a closed, orientable hyperbolic 3-manifold with $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 4$, and if the image of the cup product map $H^1(M; \mathbb{Z}_2) \otimes H^1(M; \mathbb{Z}_2) \rightarrow H^2(M; \mathbb{Z}_2)$ has dimension at most 1, then M has volume greater than 3.08. The proofs of these geometric results involve new topological results relating the Heegaard genus of a closed Haken manifold M to the Euler characteristic of the kishkes of the complement of an incompressible surface in M .

1. INTRODUCTION

If S is a properly embedded surface in a compact 3-manifold M , let $M \setminus \setminus S$ denote the manifold which is obtained by cutting along S ; it is homeomorphic to the complement in M of an open regular neighborhood of S .

The topological theme of this paper is that the bounded manifold obtained by cutting a topologically complex closed simple Haken 3-manifold along a suitably chosen incompressible surface $S \subset M$ will also be topologically complex. Here the “complexity” of M is measured by its Heegaard genus, and the “complexity” of $M \setminus \setminus S$ is measured by the absolute value of the Euler characteristic of its “kishkes” (see Definitions 1.1 below).

Our topological theorems have geometric consequences illustrating a longstanding theme in the study of hyperbolic 3-manifolds — that the volume of a hyperbolic 3-manifold reflects its topological complexity. We obtain lower bounds for volumes of closed and one-cusped hyperbolic manifolds with sufficient topological complexity, extending work of Culler and Shalen along the same lines. Here “topological complexity” is measured in terms of the mod 2 first homology, or the mod 2 cohomology ring.

Definitions 1.1. We shall say that a 3-manifold M is *simple* if

- (i) M is compact, connected, orientable, irreducible and boundary-irreducible;
- (ii) no subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$; and
- (iii) M is not a closed manifold with finite fundamental group.

Partially supported by NSF grants DMS-0204142 and DMS-0504975.

Partially supported by NSF grant DMS-0703749.

Partially supported by NSF grants DMS-0204142 and DMS-0504975.

Let X be a simple 3-manifold with $\partial X \neq \emptyset$. According to [10] or [11], the characteristic submanifold Σ_X of X is well-defined up to isotopy, and each component of Σ_X is either an I -bundle meeting ∂X in its horizontal boundary (see Definition 3.1), or a solid torus meeting ∂X in a collection of disjoint annuli that are homotopically non-trivial in X . We define $\text{kish}(X)$ (the “kishkes” of X , or “guts” in the terminology of [2]) to be the union of all components of $\overline{X - \Sigma_X}$ that have negative Euler characteristic. The components of $\text{kish}(X) \cap \Sigma_X$ are essential annuli in X .

If X is a compact 3-manifold whose components are all bounded and simple and if X_1, \dots, X_k denote the components of X , we define $\text{kish}(X) = \text{kish}(X_1) \cup \dots \cup \text{kish}(X_k) \subset X$.

Definition 1.2. Let g be an integer ≥ 2 , let h be a positive real number, and let M be an orientable, irreducible 3-manifold. We shall say that M is (g, h) -small if every connected closed incompressible surface in M has genus at least h and every separating connected closed incompressible surface in M has genus at least g .

We shall denote the Heegaard genus of a 3-manifold Q by $\text{Hg}(Q)$.

Theorem 5.8. *Suppose M is a closed, simple 3-manifold containing a separating connected closed incompressible surface of some genus g , that $\text{Hg}(M) \geq g + 4$, and that M is $(g, \frac{g}{2} + 1)$ -small. Then M contains a separating connected closed incompressible surface S of genus g satisfying at least one of the following conditions:*

- (1) *at least one component of $M \setminus S$ is acylindrical; or*
- (2) *for each component B of $M \setminus S$ we have $\text{kish}(B) \neq \emptyset$.*

The key idea in the proof is an organizing principle for cylinders properly embedded in the complement of a separating connected closed incompressible surface. This is discussed in Sections 4 and 5. We apply Theorem 5.8 in conjunction with the theorem below concerning nonseparating surfaces, which is proved in Section 3. For a manifold M with (possibly empty) boundary, let $\chi(M)$ denote the Euler characteristic of M , and let $\bar{\chi}(M) = -\chi(M)$.

Theorem 3.2. *Let M be a closed, simple 3-manifold containing a nonseparating connected closed incompressible surface S of genus g . Suppose that $\bar{\chi}(\text{kish}(M \setminus S)) < 2g - 2$, and that M is $(2g - 1, g)$ -small. Then $\text{Hg}(M) \leq 2g + 1$.*

In a closed, simple 3-manifold, every connected closed incompressible surface has genus at least 2. Thus any such manifold is $(2, 2)$ -small. Hence applying Theorems 5.8 and 3.2 to a manifold containing an embedded surface of genus 2, we will easily obtain the following corollary.

Corollary 5.9. *Suppose that M is a closed, simple 3-manifold which contains a connected closed incompressible surface of genus 2, and that $\text{Hg}(M) \geq 6$. Then M contains a connected closed incompressible surface S of genus 2 such that either $M \setminus S$ has an acylindrical component, or $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$.*

This corollary will suffice for the geometric applications in this paper. In a future paper, we will apply Theorems 3.2 and 5.8 to the case of a genus 3 surface.

In combination with work of Agol-Storm-Thurston [2] and Kojima-Miyamoto [12], Corollary 5.9 implies the following volume bound for sufficiently complex hyperbolic Haken manifolds.

Theorem 6.5. *Let M be a closed, orientable hyperbolic 3-manifold containing a closed connected incompressible surface of genus 2, and suppose that $\text{Hg}(M) \geq 6$. Then M has volume greater than 6.45.*

Theorem 6.5 implies Theorems 6.7 and 6.8 below, which extend earlier work of Culler-Shalen.

Theorem 6.7. *Let M be a complete, finite-volume, orientable hyperbolic 3-manifold having exactly one cusp, and suppose that*

$$\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 6.$$

Then M has volume greater than 5.06.

Theorem 6.5 is an improvement on [7, Proposition 10.1]. There, the stronger lower bound of 7 on the dimension of \mathbb{Z}_2 -homology gives only a conditional conclusion: either the volume bound above holds, or M contains an embedded connected closed incompressible surface of genus 2. The weakening of the lower bound on the dimension of homology in the hypothesis is made possible by the results of [6]. In the case where M contains an embedded connected closed incompressible surface of genus 2 and $H_1(M; \mathbb{Z}_2)$ has dimension at least 6, a Dehn filling argument combined with Theorem 6.5 gives a better volume bound of 6.45.

Theorem 6.8. *Let M be a closed, orientable hyperbolic 3-manifold with*

$$\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 4$$

and suppose that the image of the cup product map $H^1(M; \mathbb{Z}_2) \otimes H^1(M; \mathbb{Z}_2) \rightarrow H^2(M; \mathbb{Z}_2)$ has dimension at most 1. Then M has volume greater than 3.08.

Theorem 6.8 should be compared with Theorem 1.2 of [6], which gives the same conclusion under the hypothesis that the \mathbb{Z}_2 -homology of M has dimension at least 6, and with no restriction on cup product. As with that theorem, the proof of Theorem 6.8 uses the fact that if $\pi_1(M)$ is 3-free, M has volume greater than 3.08 (see Corollary 9.3 of [1]). If $\pi_1(M)$ has a 3-generator subgroup G which is not free, the homological hypotheses of Theorem 6.8 ensure that M has a two-sheeted cover \tilde{M} to which G lifts, with $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 6$. Then Theorem 1.1 of [6] implies that \tilde{M} contains a connected closed incompressible surface of genus 2, and Theorem 6.5 implies that \tilde{M} has volume greater than 6.45, hence that M has volume greater than 3.22.

2. TOPOLOGICAL PRELIMINARIES

In general we will follow [9] for standard terminology concerning 3-manifolds. (This includes, for example, the terms “irreducible” and “boundary-irreducible” which were used in the

introduction.) Here we will explain a few special conventions and collect some preliminary results used throughout this paper.

We will work in the PL category in Sections 2–5, and in the smooth category in Section 6. The only result from the earlier sections quoted in Section 6 is Corollary 5.9, and the smooth version of this result follows from the PL version. We will also use, generally with explicit mention, the well-known fact that a closed, orientable hyperbolic 3-manifold is simple.

In Sections 2–5 we will use the following conventions concerning regular neighborhoods. Let K be a compact polyhedron in a PL n -manifold M . We define a *semi-regular neighborhood* of K in M to be a neighborhood of K which is a compact PL submanifold of M and admits a polyhedral collapse to K . We define a *regular neighborhood* of K in M to be a semi-regular neighborhood N of K in M such that $N \cap \partial M$ is a semi-regular neighborhood of $K \cap \partial M$ in ∂M .

Let Y be a subset of a topological space X , and suppose that X and Y are locally path connected. We will say that Y is π_1 -*injective* in X if whenever A and B are components of X and Y respectively, such that $B \subset A$, the inclusion homomorphism $\pi_1(B) \rightarrow \pi_1(A)$ is injective. We will refer to $\bar{Y} \cap \overline{X - Y}$ as the *frontier* of Y in X .

A closed orientable surface S in the interior of an orientable 3-manifold M will be termed *incompressible* if S is π_1 -injective in M and no component of S is a sphere. We shall not use the term “incompressible” for bounded surfaces.

We follow the conventions of [16] regarding Heegaard splittings and compression bodies. The following standard fact is a direct consequence of the definitions.

Lemma 2.1. *Let Q be an orientable 3-manifold with boundary, and suppose \mathcal{S} is a Heegaard surface in Q .*

- (1) *Let Q' be obtained from Q by attaching a 2-handle to a component of ∂Q . Then \mathcal{S} is a Heegaard surface in Q' .*
- (2) *Let Q' be obtained from Q by attaching a handlebody to Q along a component of ∂Q . Then \mathcal{S} is a Heegaard surface in Q' .*

□

The lemma below is also standard, and will be used in Sections 3 and 5.

Lemma 2.2. *Let Q be an orientable 3-manifold with boundary, and suppose \mathcal{S} is a Heegaard surface in Q of genus g . Let Q' be obtained from Q by adding a 1-handle with both attaching disks in the same component of $Q - \mathcal{S}$. Then $\text{Hg}(Q') \leq g + 1$.*

Proof. By definition we have $Q = C_1 \cup C_2$ where C_1 and C_2 are compression bodies such that $\partial_+ C_1 = \mathcal{S} = \partial_+ C_2$ and $C_1 \cap C_2 = \mathcal{S}$. After relabeling we may assume that Q' is obtained by attaching a 1-handle H to $\partial_- C_1$. We may write C_1 as $(\mathcal{S} \times I) \cup \mathcal{D} \cup \mathcal{T}$ where \mathcal{S} is identified with $\mathcal{S} \times \{1\}$, \mathcal{D} is a union of disjoint 2-handles attached along annuli in $\mathcal{S} \times \{0\}$, and \mathcal{T} is a union of 3-handles. Since $\mathcal{D} \cap \partial C_1$ is a union of disjoint disks, there is an ambient isotopy

of C_1 which is constant on \mathcal{S} and which moves the two attaching disks of H so that they are disjoint from \mathcal{D} . We may thus assume that the attaching disks for H are contained in $\mathcal{S} \times \{0\}$.

Let N_0 be a regular neighborhood in H of its core. We have $N_0 \cap \partial_- C_1 = (E \times \{0\}) \cup (E' \times \{0\})$, where E and E' are disjoint disks in $\mathcal{S} = \partial_+ C_1$. Let $N = N_0 \cup (E \times I) \cup (E' \times I)$, so that $E = E \times \{1\}$ and $E' = E' \times \{1\}$ are contained in ∂N . Set

$$\mathcal{S}' = \overline{\mathcal{S} - (E \cup E')} \cup \overline{\partial N - (E \cup E')}.$$

The surface \mathcal{S}' has genus $g + 1$ by construction. To complete the proof we will show that it is a Heegaard surface for $Q \cup H$.

Let $P = (\overline{\mathcal{S} - (E \cup E')} \times I) \cup \overline{H - N_0}$ and set $C'_1 = P \cup \mathcal{D}$. Note that P is a semi-regular neighborhood of \mathcal{S}' in C'_1 , and hence homeomorphic to $\mathcal{S}' \times I$. The attaching annuli of the 2-handles in \mathcal{D} lie in the component of ∂P which is disjoint from \mathcal{S}' . It follows that C'_1 is a compression body with $\partial_+ C'_1 = \mathcal{S}'$.

Next let $C'_2 = C_2 \cup N$. From the dual description of C_2 as $(\partial_- C_2) \times I$ with a collection of 1-handles attached, it follows that $C_2 \cup N$ is a compression body with $\partial_+ C'_2 = \mathcal{S}'$.

By construction we have $Q' = C'_1 \cup C'_2$ and $C'_1 \cap C'_2 = \partial_+ C'_1 = \partial_+ C'_2 = \mathcal{S}'$. Thus \mathcal{S}' is indeed a Heegaard surface for Q' . \square

The following relatively straightforward result will be used in Sections 3 and 5.

Proposition 2.3. *Let $g \geq 2$ be an integer. Let M be an irreducible, orientable 3-manifold which is $(g, \frac{g+1}{2})$ -small. Let V be a compact, connected, irreducible 3-dimensional submanifold of M which is π_1 -injective. Suppose that either*

- (i) $\bar{\chi}(V) < g - 1$, or
- (ii) $\bar{\chi}(V) \leq g - 1$ and V is boundary-reducible.

Then V is a handlebody.

Proof. Choose a properly embedded (possibly empty) submanifold \mathcal{D} of V such that

- (1) each component of \mathcal{D} is an essential disk,
- (2) no two components of \mathcal{D} are parallel, and
- (3) \mathcal{D} is maximal among all properly embedded submanifolds of V satisfying (1) and (2).

(Since V is irreducible, (1) and (2) imply that the components of $\partial \mathcal{D}$ are non-trivial and pairwise non-parallel simple closed curves in ∂V ; hence a submanifold \mathcal{D} satisfying (1) and (2) has at most $3\bar{\chi}(V)$ components, and hence a maximal submanifold with these properties exists.)

Let N be a regular neighborhood of \mathcal{D} in V , and set $Q = \overline{V - N}$. In order to complete the proof it suffices to show that every component of Q is a ball.

Let us denote by B_1, \dots, B_ν the components of Q that are balls, and by R_1, \dots, R_k the remaining components of Q . A priori we have $k \geq 0$ and $\nu \geq 0$. We must show that $k = 0$.

If Q contains an essential disk D we may assume after an isotopy that $D \cap N = \emptyset$; then $D \cup \mathcal{D}$ is a properly embedded submanifold of V satisfying (1) and (2), a contradiction to the maximality of \mathcal{D} . This shows that Q is boundary-irreducible, so that ∂Q is π_1 -injective in Q . But Q is π_1 -injective in V since every component of \mathcal{D} is a disk, and V is π_1 -injective in M by hypothesis. Hence:

2.3.1. *∂Q is π_1 -injective in M .*

The manifold Q is obtained from the irreducible V by splitting along a collection of disjoint properly embedded disks. Hence:

2.3.2. *Each component of Q is irreducible.*

It follows from 2.3.2 that each R_i is irreducible. Since by definition no R_i is a ball, we deduce:

2.3.3. *No boundary component of any R_i can be a sphere.*

Let n denote the number of components of \mathcal{D} , and observe that $\bar{\chi}(Q) = \bar{\chi}(V) - n$. Next we note that by properties (1) and (2) of \mathcal{D} , each component of Q which is a ball must contain at least three components of the frontier F of N in V . Since each component of N contains exactly two components of F , the number ν of components of Q that are balls is at most $2n/3$. Hence

$$\begin{aligned} \bar{\chi}(V) - n &= \bar{\chi}(Q) \\ &= -\nu + \sum_{i=1}^k \bar{\chi}(R_i) \\ &\geq -(2n/3) + \sum_{i=1}^k \bar{\chi}(R_i), \end{aligned}$$

so that

$$(2.3.4) \quad \sum_{i=1}^k \bar{\chi}(R_i) \leq \bar{\chi}(V) - (n/3).$$

If alternative (i) of the hypothesis holds we have $\bar{\chi}(V) < g - 1$ and $n \geq 0$; if alternative (ii) holds, we have $\bar{\chi}(V) \leq g - 1$ and $n > 0$. Thus in either case, (2.3.4) implies that

$$(2.3.5) \quad \sum_{i=1}^k \bar{\chi}(R_i) < g - 1.$$

On the other hand, 2.3.3 implies that $\bar{\chi}(R_i) \geq 0$ for each i with $1 \leq i \leq k$. In view of (2.3.5) it follows that

$$(2.3.6) \quad \bar{\chi}(R_i) < g - 1$$

for each i with $1 \leq i \leq k$.

We now proceed to the proof that $k = 0$. Suppose that $k \geq 1$, and consider the manifold R_1 . By 2.3.1, the boundary of R_1 is π_1 -injective in M . By 2.3.3, no component of ∂R_1 is a sphere. Hence every component of ∂R_1 is incompressible in M . Furthermore, we have

$$\bar{\chi}(\partial R_1) = 2\bar{\chi}(R_1) < 2g - 2$$

by (2.3.6). If ∂R_1 is connected, it is a separating connected closed incompressible surface with $\bar{\chi}(\partial R_1) < 2g - 2$, so that its genus is strictly less than g . This contradicts the hypothesis.

Now suppose that ∂R_1 is disconnected. Since $\bar{\chi}(\partial R_1) < 2g - 2$ and no component of ∂R_1 is a sphere, we have $\bar{\chi}(S) < g - 1$ for some component S of ∂R_1 . This means that S is a connected closed incompressible surface of genus strictly less than $\frac{g+1}{2}$. Again we have a contradiction to the hypothesis. \square

3. NON-SEPARATING SURFACES

The purpose of this section is to prove Theorem 3.2, which was stated in the introduction.

Definition 3.1. If a 3-manifold X has the structure of an I -bundle over a surface T and $p : X \rightarrow T$ is the bundle projection, we will call $\partial_v X \doteq p^{-1}(\partial T)$ the *vertical* boundary of X and $\partial_h X \doteq \overline{\partial X} - \overline{\partial_v X}$ the *horizontal* boundary of X .

Note that $\partial_v X$ inherits the structure of an I -bundle over ∂T , and $\partial_h X$ the structure of a ∂I -bundle over T , from the original I -bundle structure on X .

Theorem 3.2. *Let M be a closed, simple 3-manifold containing a nonseparating connected closed incompressible surface S of genus g . Suppose that $\bar{\chi}(\text{kish}(M \setminus S)) < 2g - 2$, and that M is $(2g - 1, g)$ -small. Then $\text{Hg}(M) \leq 2g + 1$.*

Proof. Let M and S be as in the statement of the theorem. Set $M' = M \setminus S$, and note that $\bar{\chi}(M') = 2g - 2$. Since by hypothesis we have $\bar{\chi}(\text{kish}(M')) < 2g - 2$, the characteristic submanifold of M' has a component X which is an I -bundle over a surface with negative Euler characteristic. We identify M' with $\overline{M - N}$, where N is a regular neighborhood of S in M ; we then have $X \subset \overline{M - N} \subset M$. We set $\Sigma = N \cup X \subset M$. Since the horizontal boundary of X has Euler characteristic $2\chi(X)$, we have

$$\bar{\chi}(\Sigma) = \bar{\chi}(N) + \bar{\chi}(X) - 2\bar{\chi}(X) = 2g - 2 - \bar{\chi}(X).$$

Since $\chi(X) < 0$, it follows that $\bar{\chi}(\Sigma) < 2g - 2$.

Set $K = \overline{M - \Sigma} \subset \overline{M - N} = M'$. (It may happen that $K = \emptyset$.) Since $\partial K = \partial \Sigma$, we have $\bar{\chi}(K) = \bar{\chi}(\Sigma)$, and hence

$$(3.2.1) \quad \bar{\chi}(K) < 2g - 2.$$

Since the frontier components of K in M' are essential annuli, K is π_1 -injective in M' . The incompressibility of S implies that $M' = \overline{M - N}$ is π_1 -injective in M . Hence:

3.2.2. *K is π_1 -injective in M .*

Note also that M' is irreducible because the surface S is incompressible in the irreducible 3-manifold M . The manifold K is a union of components of the manifold obtained by splitting M' along a collection of disjoint properly embedded annuli. Hence:

3.2.3. *Every component of K is irreducible.*

Since each component of K contains a component of the frontier of the characteristic submanifold of M' , which is an essential annulus in M' , no component of K is a ball. In view of 3.2.3 it follows that no component of ∂K is a sphere. Hence:

3.2.4. *Every component of K has non-positive Euler characteristic.*

Now consider any component V of K . Set $V' = K - V$. It follows from 3.2.4 that $\bar{\chi}(V') \geq 0$. We have $\bar{\chi}(V) = \bar{\chi}(K) - \bar{\chi}(V')$, and hence by 3.2.1

$$(3.2.5) \quad \bar{\chi}(V) < 2g - 2.$$

By hypothesis M is $(2g - 1, g)$ -small. Since $g = \frac{(2g-1)+1}{2}$, this means that M is $(2g - 1, \frac{(2g-1)+1}{2})$ -small. In view of 3.2.2, 3.2.3 and 3.2.5, case (i) of the hypothesis of Proposition 2.3 holds with $2g - 1$ playing the role of g . Proposition 2.3 therefore implies that V is a handlebody.

Thus we have shown:

3.2.6. *Every component of $K = \overline{M - \Sigma}$ is a handlebody.*

We now turn to the estimation of $\text{Hg}(M)$. First note that since N is a trivial I -bundle over a surface of genus g , it can be obtained from a handlebody J of genus $2g$ by adding a 2-handle. The boundary \mathcal{S} of a collar neighborhood of ∂J in J is a Heegaard surface of genus $2g$ in J . Hence by assertion (1) of Lemma 2.1, \mathcal{S} is a Heegaard surface in N . Note that ∂N is contained in a single component of $N - \mathcal{S}$.

On the other hand, recall that $\Sigma = N \cup X$, where X is an I -bundle over a connected surface T and $N \cap X = \partial_h X$ is the horizontal boundary of X . Let $E \subset T$ be a disk such that for each boundary component c of T , the set $E \cap c$ is a non-empty union of disjoint arcs in c . Let $p : X \rightarrow T$ denote the bundle projection, and set $Y = p^{-1}(E)$. Then Y inherits the structure of a (necessarily trivial) I -bundle over E , and $Y \cap N = Y \cap \partial_h X$ is the horizontal boundary of Y , consisting of two disks. Thus the set Y may be thought of as a 1-handle attached to the submanifold N . Since \mathcal{S} is a genus- $2g$ Heegaard surface in N , and ∂N is contained in a single component of $N - \mathcal{S}$, it follows from Lemma 2.2 that $\text{Hg}(N \cup Y) \leq 2g + 1$.

Next, note that each component of $\overline{(\partial T) - E}$ is an arc, and hence that each component of the set $\mathcal{D} \doteq p^{-1}(\overline{(\partial T) - E})$ is a disk. Note also that $\mathcal{D} \cap (N \cup Y) = \partial \mathcal{D}$. Hence if R denotes a regular neighborhood of \mathcal{D} relative to $\overline{X - Y}$, saturated in the fibration of X , the manifold $N \cup Y \cup R$ is obtained from $N \cup Y$ by adding finitely many 2-handles. By assertion (1) of Lemma 2.1 it follows that

$$\text{Hg}(N \cup Y \cup R) \leq \text{Hg}(N \cup Y) \leq 2g + 1.$$

Finally, note that each component of $\overline{M - (N \cup Y \cup R)}$ is either

- (a) a component of $\overline{X - (Y \cup R)}$ or
- (b) a component of $\overline{M - \Sigma}$.

Each component of type (a) is a sub-bundle of X over a bounded subsurface, and is therefore a handlebody. Each component of type (b) is a handlebody by virtue of 3.2.6. Since each component of $\overline{M - (N \cup Y \cup R)}$ is a handlebody, it now follows from assertion (2) of Lemma 2.1 that

$$\text{Hg}(M) \leq \text{Hg}(N \cup Y \cup R) \leq 2g + 1.$$

□

4. ANNULUS BODIES AND SHALLOW MANIFOLDS

In the next two sections, we develop an organizing principle for cylinders properly embedded in the complement of a separating connected closed incompressible surface.

Definition 4.1. Let Y be a compact, connected 3-manifold, and let S be a (possibly disconnected) closed, 2-dimensional submanifold of ∂Y . We shall say that Y is an *annulus body* relative to S if there is a properly embedded annulus $A \subset Y$ with $\partial A \subset S$, such that Y is a semi-regular neighborhood of $S \cup A$.

Lemma 4.2. *Let Y be a compact, connected 3-manifold, and let $S \subset \partial Y$ be a closed 2-manifold. If Y is an annulus body relative to S , then Y is also an annulus body relative to $(\partial Y) - S$. Furthermore, we have $\bar{\chi}(Y) = \bar{\chi}(S) = \bar{\chi}((\partial Y) - S)$.*

Proof. We set $T = (\partial Y) - S$.

By the definition of an annulus body, Y is a semi-regular neighborhood of $S \cup A$ for some properly embedded annulus $A \subset Y$ with $\partial A \subset S$. Let R be a regular neighborhood of A in Y . Then there is a PL homeomorphism $j : S^1 \times [-1, 1] \times [-1, 1] \rightarrow R$ such that $j(S^1 \times \{0\} \times [-1, 1]) = A$ and $j(S^1 \times [-1, 1] \times \{-1, 1\}) = R \cap S$. Let B denote the annulus $j(S^1 \times [-1, 1] \times \{0\})$. Set $Q = \overline{Y - R}$, and let N be a regular neighborhood of $Q \cap S$ in Q , chosen small enough so that $N \cap B = \emptyset$. Set $Y' = N \cup R$. Then Y' is a compact 3-manifold and $S \subset \partial Y'$. If we set $T' = (\partial Y') - S$, then the annulus B is properly embedded in Y' and $\partial B \subset T'$. Furthermore, Y' is a semi-regular neighborhood of $T' \cup B$, and hence Y' is an annulus body relative to T' .

On the other hand, Y and $Y' \subset Y$ are both semi-regular neighborhoods of $S \cup A$, and $Y' \cap \partial Y = S$. Hence $\overline{Y - Y'}$ is a collar neighborhood of $T \subset \partial Y$ in Y . In particular the pairs (Y, T) and (Y', T') are PL homeomorphic, and so Y is an annulus body relative to T .

To prove the second assertion, we note that since Y and $S \cup A$ are homotopy equivalent, we have $\chi(Y) = \chi(S \cup A)$; and that since A and $A \cap S = \partial A$ have Euler characteristic 0, we have $\chi(S \cup A) = \chi(S)$. This proves that $\bar{\chi}(S) = \bar{\chi}(Y)$. Since we have shown that Y is also an annulus body relative to T , we may substitute T for S in the last equality and conclude that $\bar{\chi}(T) = \bar{\chi}(Y)$. □

Definition 4.3. Let Z be a compact, connected, orientable 3-manifold, and let $S \subset \partial Z$ be a closed surface. We will say that Z is *shallow* relative to S if Z may be written in the form $Z = Y \cup J$, where $Y \supset S$ and J are compact 3-dimensional submanifolds of Z such that

- (1) each component of J is a handlebody,
- (2) Y is an annulus body relative to S ,
- (3) $Y \cap J = \partial J$, and
- (4) ∂J is a union of components of $(\partial Y) - S$.

(The submanifold J may be empty.)

Lemma 4.4. *Let Q be a compact orientable 3-manifold, let $Z \subset Q$ be a compact submanifold whose frontier S is a connected closed surface in $\text{int } Q$, and suppose that Z is shallow relative to S . Then $\text{Hg}(Q) \leq 1 + \text{Hg}(\overline{Q - Z})$.*

Proof. We set $Q_0 = \overline{Q - Z}$ and $g = \text{Hg}(Q_0)$. We write $Z = Y \cup J$, where Y and J satisfy conditions (1)–(4) of Definition 4.3. Since Y is an annulus body relative to S , it follows from Definition 4.1 that there is a properly embedded annulus $A \subset Y$ with $\partial A \subset S$, such that Y is a semi-regular neighborhood, relative to Y itself, of $S \cup A$.

Let α denote a co-core of the annulus A , and fix a regular neighborhood h of α in Y such that $h \cap A$ is a regular neighborhood of α in A . The manifold $Q_0 \cup h$ is obtained from Q_0 by attaching a 1-handle that has both its attaching disks in the component S of ∂Q_0 . Hence it follows from Lemma 2.2 that $\text{Hg}(Q_0 \cup h) \leq 1 + \text{Hg}(Q_0) = 1 + g$.

The disk $D = \overline{A - (h \cap A)}$ is properly embedded in the manifold $\overline{Y - h}$. Hence if R denotes a regular neighborhood of D relative to $\overline{Y - h}$, the manifold $X_0 \doteq Q_0 \cup h \cup R$ is obtained from $Q_0 \cup h$ by attaching a 2-handle. It therefore follows from assertion (1) of Lemma 2.1 that $\text{Hg}(X_0) \leq \text{Hg}(Q_0 \cup h) \leq 1 + g$. But $X \doteq Q_0 \cup Y$ is a semi-regular neighborhood of X_0 relative to X itself, and is therefore homeomorphic to X_0 . Hence $\text{Hg}(X) \leq 1 + g$.

We have $Q = X \cup J$. In view of Conditions (1), (3) and (4) of Definition 4.3, it follows that each component of $\overline{Q - X}$ is a handlebody whose boundary is contained in ∂X . From assertion (2) of Lemma 2.1 we deduce that $\text{Hg}(Q) \leq \text{Hg}(X) \leq 1 + g$. \square

Lemma 4.5. *Suppose that Z is a compact, connected, orientable 3-manifold, that ∂Z is connected, and that Z is shallow relative to ∂Z . Let g denote the genus of ∂Z . Then $\text{Hg}(Z) \leq g + 1$.*

Proof. Let N be a boundary collar for Z . Then N has a Heegaard splitting of genus g , the frontier S of N in Z is connected, and $\overline{Z - N}$ is shallow relative to S . The result therefore follows upon applying Lemma 4.4, with Z and $\overline{Z - N}$ playing the respective roles of Q and Z in that lemma. \square

Lemma 4.6. *Let $g \geq 2$ be an integer. Let Z be a compact, orientable 3-manifold having exactly two boundary components S_0 and S_1 , both of genus g . Then Z is shallow relative to S_0 if and only if either*

- (i) Z is an annulus body relative to S_0 , or
- (ii) there is a solid torus $K \subset Z$ such that $K \cap \partial Z$ is an annulus contained in S_1 , and the pair $(\overline{Z - K}, S_0)$ is homeomorphic to $(S_0 \times I, S_0 \times \{0\})$.

Proof. If alternative (i) holds then Z is shallow relative to S_0 : it suffices to take $Y = Z$ and $J = \emptyset$ in Definition 4.3. If alternative (ii) holds, let J be a regular neighborhood in $\text{int } K$ of a core curve of K and set $Y = \overline{Z - J}$. Then Y is an annulus body relative to S_1 . (The annulus A appearing in Definition 4.1 is bounded by two parallel simple closed curves.) It now follows from Definition 4.3 that Z is shallow relative to S_0 .

Conversely, suppose that Z is shallow relative to S_0 . Let us write $Z = Y \cup J$, where Y and J satisfy conditions (1)–(4) of Definition 4.3, with S_0 playing the role of S . Set $T = (\partial Y) - S_0$. Since Y is an annulus body relative to S_0 , it follows from Lemma 4.2 that Y is an annulus body relative to T . This means that Y is a semi-regular neighborhood of $T \cup A$, where A is an annulus with $A \cap T = \partial A$. Since Y is connected it follows that T has at most two components.

The conditions of Definition 4.3 imply that T is the disjoint union of ∂J with S_1 . Since T has at most two components and S_1 has exactly one component, ∂J has at most one component. If $\partial J = \emptyset$ then $J = \emptyset$, i.e. $Z = Y$. This implies alternative (i) of the present lemma.

Now consider the case in which ∂J has exactly one component. In this case J is a single handlebody, and ∂J and S_1 are the components of T . According to Lemma 4.2 we have

$$2g - 2 = \bar{\chi}(S_0) = \bar{\chi}(T) = \bar{\chi}(S_1) + \bar{\chi}(\partial J).$$

But since S_1 is a surface of genus g we have $\bar{\chi}(S_1) = 2g - 2$, and hence $\bar{\chi}(J) = 0$. Thus J is a solid torus.

Now Y is a semi-regular neighborhood of $T \cup A = S_1 \cup A \cup \partial J$. Since Y is connected, A must have one boundary component in S_1 and one in ∂J . Let R be a regular neighborhood of A in Y . Then R is a solid torus meeting S_1 and ∂J , respectively, in regular neighborhoods of the simple closed curves $A \cap S_1$ and $A \cap \partial J$. Let $K = J \cup R$. Since J is a solid torus and $A \cap J$ is parallel in R to its core, K is a solid torus. Furthermore, $K \cap S_1 = R \cap S_1$ is an annulus.

Now set $Q = \overline{Y - R} = \overline{Z - K}$. If N is a regular neighborhood in Q of $Q \cap T$, then $Y' = N \cup R$ is a semiregular neighborhood of $T \cup A$ contained in Y . Therefore $\overline{Y - Y'}$ is a collar neighborhood of S_0 in Y ; that is, the pairs $(\overline{Y - Y'}, S_0)$ and $(S_0 \times I, S_0 \times \{0\})$ are homeomorphic. But by the definition of N , the pair $(\overline{Y - Y'}, S_0) = (\overline{Q - N}, S_0)$ is homeomorphic to $(Q, S_0) = (\overline{Z - K}, S_0)$, and alternative (ii) of the present lemma holds in this case. \square

Lemma 4.7. *Let $g \geq 2$ be an integer. Let Z be a compact, orientable 3-manifold having exactly two boundary components S_0 and S_1 , both of genus g . Then Z is shallow relative to S_0 if and only if it is shallow relative to S_1 .*

Proof. By symmetry it suffices to show that if Z is shallow relative to S_0 then it is shallow relative to S_1 . In view of Lemma 4.6, it suffices to show that if one of the alternatives (i) and (ii) of that lemma holds, then it still holds when S_0 is replaced by S_1 . For alternative (i) this follows from Lemma 4.2. Now suppose that alternative (ii) of Lemma 4.6 holds. Let c be a core curve of the annulus $\overline{\partial K - (K \cap \partial Z)}$. Since $(\overline{Z - K}, S_0)$ is homeomorphic to $(S_0 \times I, S_0 \times \{0\})$, there is a properly embedded annulus $\alpha \subset \overline{Z - K}$ joining c to a simple closed curve in S_0 . Let B be a regular neighborhood of $K \cup \alpha$ in Z . Set $P = \overline{Z - B}$, choose a regular neighborhood N of $P \cup S_1$ in Z , and set $K' = \overline{Z - N}$. Then K' is a solid torus, $K' \cap \partial Z$ is an annulus contained in S_0 , and the pair $(\overline{Z - K'}, S_1) = (N, S_1)$ is homeomorphic to $(S_1 \times I, S_1 \times \{0\})$. \square

5. SEPARATING SURFACES

5.1. In this section we will use the theory of books of I -bundles as developed in [1]. We recall the definition here, in a slight paraphrase of the form given in [1].

A *book of I -bundles* is a triple $\mathcal{W} = (W, \mathcal{B}, \mathcal{P})$, where W is a (possibly disconnected) compact, orientable 3-manifold, and $\mathcal{B}, \mathcal{P} \subset W$ are submanifolds such that

- each component of \mathcal{B} is a solid torus;
- \mathcal{P} is an I -bundle over a (possibly disconnected) 2-manifold, and every component of \mathcal{P} has Euler characteristic ≤ 0 ;
- $W = \mathcal{B} \cup \mathcal{P}$;
- $\mathcal{B} \cap \mathcal{P}$ is the vertical boundary of \mathcal{P} ;
- $\mathcal{B} \cap \mathcal{P}$ is π_1 -injective in \mathcal{B} ; and
- each component of \mathcal{B} meets at least one component of \mathcal{P} .

As in [1], we shall denote W , \mathcal{B} and \mathcal{P} by $|\mathcal{W}|$, $\mathcal{B}_{\mathcal{W}}$ and $\mathcal{P}_{\mathcal{W}}$ respectively. The components of $\mathcal{B}_{\mathcal{W}}$ will be called *bindings* of \mathcal{W} , and the components of $\mathcal{P}_{\mathcal{W}}$ will be called its *pages*. The submanifold $\mathcal{B} \cap \mathcal{P}$, whose components are properly embedded annuli in W , will be denoted $\mathcal{A}_{\mathcal{W}}$.

An important observation, which follows from the definitions, is that if W is a simple 3-manifold with $\text{kish } W = \emptyset$, then $W = |\mathcal{W}|$ for some book of I -bundles \mathcal{W} .

Lemma 5.2. *If \mathcal{W} is any connected book of I -bundles then $\mathcal{A}_{\mathcal{W}}$ is π_1 -injective in $|\mathcal{W}|$. Furthermore, $|\mathcal{W}|$ is an irreducible 3-manifold.*

Proof. If A is any component of $\mathcal{A} = \mathcal{A}_{\mathcal{W}}$, then A lies in the frontier of a unique component P of $\mathcal{P} = \mathcal{P}_{\mathcal{W}}$ and in a unique component B of $\mathcal{B} = \mathcal{B}_{\mathcal{W}}$. Since A is an annulus of non-zero degree in the solid torus B , it is π_1 -injective in B . It is also π_1 -injective in P , since A is a vertical boundary annulus of the I -bundle P and $\chi(P) \leq 0$. It follows that \mathcal{A} is π_1 -injective in \mathcal{B} and in \mathcal{P} and hence in $W = |\mathcal{W}|$, which is the first assertion.

To prove the second assertion, we note that \mathcal{B} is irreducible because its components are solid tori, and that \mathcal{P} is irreducible because its components are I -bundles over surfaces of

Euler characteristic ≤ 0 . Thus W contains the π_1 -injective, two-sided, properly embedded 2-manifold \mathcal{A} , and the manifold obtained by splitting W along \mathcal{A} is irreducible. It follows that W is itself irreducible. \square

Lemma 5.3. *Let M be a closed simple 3-manifold. Suppose that \mathcal{W} is a connected book of I -bundles with $W = |\mathcal{W}| \subset M$, and that ∂W is a connected incompressible surface in M . Let g denote the genus of ∂W . Suppose that M is $(g, \frac{g+1}{2})$ -small. Then W is shallow relative to ∂W .*

Proof. We first consider the degenerate case in which \mathcal{W} has no bindings, so that W is an I -bundle over a closed surface T with $\chi(T) = 2 - 2g < 0$. We choose an orientation-preserving simple closed curve $C \subset T$ and let A denote the annulus $p^{-1}(C)$, where p is the bundle projection. If Y denotes a regular neighborhood of $(\partial W) \cup A$, then $\overline{W - Y}$ is homeomorphic to an I -bundle over a bounded surface, and hence to a handlebody; hence W is shallow in this case.

Now assume that \mathcal{W} has at least one binding, so that every page is an I -bundle over a bounded surface. The sum of the Euler characteristics of the pages of \mathcal{W} is equal to $\chi(W) = 1 - g < 0$. In particular, \mathcal{W} has a page P_0 with $\chi(P_0) < 0$. Then P_0 is an I -bundle over some base surface T ; we let $p : P_0 \rightarrow T$ denote the bundle projection. We choose a component C of ∂T and set $A = p^{-1}(C) \subset \partial_v P_0$. (See Definition 3.1.) Since $\chi(T) = \chi(P_0) < 0$, there is an arc $\alpha \subset T$ such that $\partial\alpha = \alpha \cap \partial T \subset C$, and α is not parallel in T to an arc in C . Now A is a properly embedded annulus in W , and $D = p^{-1}(\alpha)$ is a properly embedded disk in P_0 . The boundary of D consists of two vertical arcs in $A \subset \partial_v P_0$, and of two properly embedded arcs in $\partial_h A_0$, each of which projects to α under the bundle projection. Since α is not parallel in T to an arc in C , each of the four arcs comprising ∂D is essential in either $\partial_h P_0$ or A . Hence ∂D is homotopically non-trivial in $A \cup \partial_h P_0$.

We set $X = W \setminus A$. We may identify P_0 with a submanifold of X . Since $\partial D \subset A \cup \partial_h P_0$, the disk D is properly embedded in X . Each component of $\partial_v P_0$ is an annulus in $\mathcal{A}_{\mathcal{W}}$ by definition. Since the frontier curves of $A \cup \partial_h P_0$ in ∂X are boundary components of such annuli, it follows from Lemma 5.2 that they are homotopically nontrivial in X . Therefore $A \cup \partial_h P_0$ is π_1 -injective in ∂X . Since ∂D is homotopically non-trivial in $A \cup \partial_h P_0$, it is homotopically non-trivial in ∂X . Hence the disk D is essential in X .

Note that ∂X may have either one or two components. In either case we have $\bar{\chi}(X) = \bar{\chi}(W) = g - 1$.

The manifold W is irreducible by Lemma 5.2, and X is obtained by splitting W along a properly embedded surface. Hence:

5.3.1. *Each component of X is irreducible.*

We shall identify X homeomorphically with $\overline{W - N}$, where N is a regular neighborhood of A in W . With this identification, it follows from Lemma 5.2 that X is π_1 -injective in W . On the other hand, the incompressibility of ∂W implies that W is π_1 -injective in M . Hence:

5.3.2. X is π_1 -injective in M .

We now claim:

5.3.3. Each component of X is a handlebody.

To prove 5.3.3, we recall that by hypothesis M is $(g, \frac{g+1}{2})$ -small. Furthermore each component of X is π_1 -injective in M by 5.3.2, and is irreducible by 5.3.1. Hence it suffices to show that one of the conditions (i) or (ii) of Proposition 2.3 holds for each component V of X .

If X is connected, then it is boundary-reducible since it contains the essential disk D . Since $\bar{\chi}(X) = g - 1$, condition (ii) of Proposition 2.3 holds with $V = X$.

Now suppose that X has two components X_0 and X_1 . Each of these is a union of pages and bindings of \mathcal{W} , and we may suppose them to be indexed so that $P_0 \subset X_0$. For $i = 0, 1$ we have $\bar{\chi}(X_i) = \sum \bar{\chi}(P)$, where P ranges over the pages contained in X_i . By the definition of a book of I -bundles, each term $\bar{\chi}(P)$ is non-negative. In particular we have $\bar{\chi}(X_1) \geq 0$, and since $\bar{\chi}(P_0) > 0$ we have $\bar{\chi}(X_0) > 0$. On the other hand, we have

$$\bar{\chi}(X_0) + \bar{\chi}(X_1) = \bar{\chi}(X) = g - 1.$$

It follows that $\bar{\chi}(X_1) < g - 1$, so that condition (i) of Proposition 2.3 holds with $V = X_1$. On the other hand, we have $\bar{\chi}(X_0) \leq g - 1$, and X_0 is boundary-reducible since it contains the essential disk D . Thus condition (i) of Proposition 2.3 holds with $V = X_0$. This proves 5.3.3.

If Y denotes a regular neighborhood of $(\partial W) \cup A$, then Y is an annulus body relative to ∂W , and $\overline{W - Y}$ is homeomorphic to X and is therefore a handlebody. By Definition 4.3 it follows that W is shallow. \square

Definition 5.4. Let M be a closed orientable 3-manifold, and let g be an integer ≥ 2 . We define a g -layering to be a finite sequence $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$, where

- n is a strictly positive integer,
- S_1, \dots, S_n are disjoint separating incompressible surfaces in M with genus g ,
- Z_0, Z_1, \dots, Z_n are the closures of the components of $M - (S_1 \cup \dots \cup S_n)$,
- $\partial Z_0 = S_1$, $\partial Z_n = S_n$, and $\partial Z_i = S_i \sqcup S_{i+1}$ for $0 < i < n$, and
- for $0 < i < n$, Z_i is shallow relative to S_i and is not homeomorphic to $S_i \times I$.

We shall call the integer n the *depth* of the g -layering $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$. We will say that a g -layering $(Z'_0, S'_1, Z'_1, \dots, S'_{n'}, Z'_{n'})$ is a (*strict*) *refinement* of $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$ if (S_1, \dots, S_n) is a (proper) subsequence of the finite sequence $(S'_1, \dots, S'_{n'})$. A g -layering will be called *maximal* if it has no strict refinement.

Lemma 5.5. *Let M be a closed orientable 3-manifold, and let g be an integer ≥ 2 . If $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$ is a g -layering, then $(Z_n, S_n, Z_{n-1}, \dots, S_1, Z_0)$ is also a g -layering. Furthermore, if $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$ is maximal, then $(Z_n, S_n, Z_{n-1}, \dots, S_1, Z_0)$ is also maximal.*

Proof. If $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$ is a g -layering, then for $0 < i < n$, since Z_i is shallow relative to S_i , it follows from Lemma 4.7 that Z_i is shallow relative to S_{i+1} . This implies the first assertion.

To prove the second assertion, suppose that $(Z_n, S_n, Z_{n-1}, \dots, S_1, Z_0)$ is not maximal, so that it has a strict refinement $(Z'_{n'}, S'_{n'}, Z'_{n'-1}, \dots, S'_1, Z'_0)$. Then $(Z'_0, S'_1, \dots, Z'_{n'-1}, S'_{n'}, Z'_{n'})$ is a g -layering according to the first assertion, and is a strict refinement of $(Z_0, S_1, \dots, Z_{n-1}, S_n, Z_n)$. Hence $(Z_0, S_1, \dots, Z_{n-1}, S_n, Z_n)$ is not maximal. \square

Lemma 5.6. *Let M be a closed orientable 3-manifold, and let g be an integer ≥ 2 . Suppose that M is $(g, \frac{g}{2} + 1)$ -small. If $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$ is a maximal g -layering, then Z_n is either shallow relative to S_n or acylindrical.*

Proof. Let us suppose that Z_n is not acylindrical. Let A be an essential annulus in Z_n . We set $X = Z_n \setminus A$. We shall identify X homeomorphically with $\overline{Z_n - N}$, where N is a regular neighborhood of A in Z_n .

If Y denotes a regular neighborhood of $(S_n) \cup A$, then Y is an annulus body relative to S_n , and $\overline{Z_n - Y}$ is ambiently isotopic to X . Hence in order to show that Z_n is shallow, it suffices to show that each component of $\overline{Z_n - Y}$ is a handlebody.

Note that ∂X has at most two components, and hence that X has at most two components. We have

$$(5.6.1) \quad g - 1 = \bar{\chi}(Z_n) = \bar{\chi}(X) = \sum_V \bar{\chi}(V),$$

where V ranges over the components of X . The essentiality of A implies that $\bar{\chi}(F) \geq 0$ for each component F of ∂X , and hence that $\bar{\chi}(V) \geq 0$ for each component V of ∂X . It therefore follows from (5.6.1) that $\bar{\chi}(V) \leq g - 1$ for each component V of X .

Since A is an essential annulus, X is π_1 -injective in Z_n . On the other hand, the incompressibility of S_n implies that Z_n is π_1 -injective in M . Hence:

5.6.2. *X is π_1 -injective in M .*

Since M is irreducible, the incompressibility of S_n implies that Z_n is irreducible. Since A is a properly embedded annulus in Z_n , we deduce:

5.6.3. *Each component of X is irreducible.*

We now claim:

5.6.4. *For each component V of X , some component of ∂V is compressible in M .*

To prove 5.6.4, we first consider the case in which ∂V is disconnected. In this case, since $\bar{\chi}(\partial V) = 2\bar{\chi}(V) \leq 2g - 2$, there is a component F of ∂V with $\bar{\chi}(F) < g$; hence the genus of F is strictly less than $\frac{g}{2} + 1$. Since M is $(g, \frac{g}{2} + 1)$ -small, the surface F must be compressible.

We next consider the case in which ∂V is connected and $\bar{\chi}(V) < g - 1$. In this case we have $\bar{\chi}(\partial V) < 2g - 2$, so that ∂V has genus strictly less than g . Furthermore, ∂V separates M . Since M is $(g, \frac{g}{2} + 1)$ -small, the surface ∂V must be compressible.

There remains the case in which ∂V is connected and $\bar{\chi}(V) = g - 1$. In this case we set $S = \partial V$ and observe that S is a separating surface of genus g . We shall assume that S is incompressible and derive a contradiction. Since $X' \doteq \overline{Z_n - Y}$ is ambiently isotopic to X , some component V' of X' is ambiently isotopic to V , and so $S' \doteq \partial V'$ is a separating connected closed incompressible surface of genus g .

We distinguish two subcases, depending on whether (a) V is the only component of X , or (b) X has a second component U . In subcase (a), the boundary components of Y are S_n and S' . Since Y is an annulus body relative to S_n , it is in particular shallow relative to S_n . Furthermore, Y cannot be homeomorphic to $S_n \times I$, because it contains the annulus A , which is essential in Z_n —and hence in Y —and has its boundary contained in S_n . It now follows from Definition 5.4 that $(Z_0, S_1, Z_1, \dots, S_n, Y, S', X')$ is a g -layering. This contradicts the maximality of $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$.

In subcase (b), it follows from (5.6.1) that $\bar{\chi}(U) = 0$. Since ∂X has at most two components, ∂U is a single torus. The simplicity of M implies that ∂U is compressible in M , and since U is π_1 -injective in M by 5.6.2, ∂U cannot be π_1 -injective in U . As U is irreducible by 5.6.3, it now follows that U is a solid torus. Hence the component U' of X' which is ambiently isotopic to U is a solid torus. According to Definition 4.3, this implies that $Z \doteq Y \cup U'$ is shallow. The boundary components of Z are S_n and S' . The shallow manifold Z cannot be homeomorphic to $S_n \times I$, because it contains the annulus A , which is essential in Z_n —and hence in Z —and has its boundary contained in S_n . It now follows from Definition 5.4 that $(Z_0, S_1, Z_1, \dots, S_n, Z, S', V')$ is a g -layering. This contradicts the maximality of $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$.

This completes the proof of 5.6.4.

Next we claim:

5.6.5. *Each component of X is boundary-reducible.*

In fact, if a component V of X were boundary-irreducible, then ∂V would be π_1 -injective in V . In view of 5.6.2 it would follow that ∂V is π_1 -injective in M . But this contradicts 5.6.4. Thus 5.6.5 is established.

We now turn to the proof that each component of X is a handlebody, which will complete the proof of the lemma.

Let V be any component of X . We have observed that $\bar{\chi}(V) \leq g - 1$. By 5.6.2, V is π_1 -injective in M , by 5.6.3 it is irreducible, and by 5.6.5 it is boundary-reducible. Since the hypothesis implies in particular that M is $(g, \frac{g+1}{2})$ -small, case (ii) of the hypothesis of Proposition 2.3 holds. It therefore follows from Proposition 2.3 that V is a handlebody. \square

Lemma 5.7. *Let M be a closed orientable 3-manifold, and let g be an integer ≥ 2 . Suppose that M is $(g, \frac{g}{2} + 1)$ -small. If $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$ is a maximal g -layering, then Z_0 is either shallow relative to S_1 or acylindrical.*

Proof. This is an immediate consequence of Lemmas 5.5 and 5.6. □

Theorem 5.8. *Suppose M is a closed, simple 3-manifold containing a separating connected closed incompressible surface of some genus g , that $\text{Hg}(M) \geq g + 4$, and that M is $(g, \frac{g}{2} + 1)$ -small. Then M contains a separating connected closed incompressible surface S of genus g satisfying at least one of the following conditions:*

- (1) *at least one component of $M \setminus S$ is acylindrical; or*
- (2) *for each component B of $M \setminus S$ we have $\text{kish}(B) \neq \emptyset$.*

Proof. It follows from the Haken finiteness theorem [9, Lemma 13.2] that the set of all depths of g -layerings in M is bounded. In particular any g -layering has a refinement which is a maximal g -layering.

By hypothesis M contains some separating connected closed incompressible surface T of genus g . If X and Y denote the closures of the components of $M - T$, then (X, T, Y) is a g -layering of depth 1. In particular M contains a g -layering, and hence contains a maximal g -layering.

Now suppose that the conclusion of Theorem 5.8 does not hold. Fix a maximal g -layering $(Z_0, S_1, Z_1, \dots, S_n, Z_n)$. Then neither Z_0 nor Z_n is acylindrical, since otherwise $S = S_1$ or S_n would satisfy conclusion (1) of the theorem. In view of 5.6 and 5.7, and the hypothesis that M is $(g, \frac{g}{2} + 1)$ -small, it follows that Z_0 and Z_n are both shallow.

For $0 < i \leq n$ we define $B_i^- = Z_0 \cup Z_1 \cup \dots \cup Z_{i-1}$ and $B_i^+ = Z_i \cup \dots \cup Z_n$. For each i , we must have either $\text{kish}(B_i^-) = \emptyset$ or $\text{kish}(B_i^+) = \emptyset$, since otherwise $S = S_i$ would satisfy conclusion (2) of the theorem. Hence by the observation made in 5.1, at least one of B_i^- or B_i^+ has the form $|\mathcal{W}|$ for some book of I -bundles \mathcal{W} . Since M is in particular $(g, \frac{g+1}{2})$ -small, it now follows from Lemma 5.3 that at least one of B_i^- or B_i^+ is shallow.

Since $B_0^- = Z_0$ is shallow, there is a largest index $k \leq n$ such that B_k^- is shallow. Since B_k^- is shallow, it follows from Lemma 4.5 that $\text{Hg}(B_k^-) \leq g + 1$. We distinguish two cases depending on whether $k < n$ or $k = n$.

If $k < n$ then B_{k+1}^- is not shallow, and hence B_{k+1}^+ is shallow. By the definition of B_{k+1}^- , the frontier of Z_k in B_{k+1}^- is the closed surface S_k , and by the definition of a g -layering, Z_k is shallow relative to S_k . We may thus apply Lemma 4.4 with $Q = B_{k+1}^-$ and $Z = Z_k$ to deduce that

$$\text{Hg}(B_{k+1}^-) \leq 1 + \text{Hg}(B_k^-) \leq g + 2.$$

Now since B_{k+1}^+ is shallow, we may again apply Lemma 4.4, this time with $Q = M$ and $Z = B_{k+1}^+$, to deduce that

$$\text{Hg}(M) \leq 1 + \text{Hg}(B_{k+1}^-) \leq g + 3.$$

This contradicts the hypothesis.

Now suppose that $k = n$. In this case, since $B_n^+ = Z_n$ is shallow, we may apply Lemma 4.4 with $Q = M$ and $Z = Z_n$ to deduce that

$$\text{Hg}(M) \leq 1 + \text{Hg}(B_n^-) \leq g + 2.$$

Once again we have a contradiction to the hypothesis. \square

Corollary 5.9. *Suppose that M is a closed, simple 3-manifold which contains a connected closed incompressible surface of genus 2, and that $\text{Hg}(M) \geq 6$. Then M contains a connected closed incompressible surface S of genus 2 such that either $M \setminus S$ has an acylindrical component, or $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$.*

Proof. Since M is simple, it is $(2, 2)$ -small.

First consider the case in which M contains a separating, connected, closed, incompressible surface of genus 2. Since M is $(2, 2)$ -small, Theorem 5.8 gives a separating connected closed incompressible surface S of genus 2 such that either at least one component of $M \setminus S$ is acylindrical, or for each component B of $M \setminus S$ we have $\text{kish}(B) \neq \emptyset$. In particular, $\text{kish}(M \setminus S)$ has at least two components. By Definition (see 1.1), each component of $\text{kish}(M \setminus S)$ has a strictly negative Euler characteristic. Hence $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$.

Now suppose that M contains no separating, connected, closed, incompressible surface of genus 2. In this case M is $(3, 2)$ -small. By hypothesis, M contains a connected, closed, incompressible surface S of genus 2, which must be non-separating. It now follows from Theorem 3.2 that $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$. \square

6. VOLUME BOUNDS

Recall that a *slope* on a torus T is an unoriented isotopy class of homotopically non-trivial simple closed curves on T . If the torus T is a boundary component of an orientable 3-manifold N , and r is a slope on T , we denote by $N(r)$ the ‘‘Dehn-filled’’ manifold obtained from the disjoint union of N with $D^2 \times S^1$ by gluing $(\partial D^2) \times S^1$ to ∂N via a homeomorphism which maps $(\partial D^2) \times \{\text{point}\}$ to a curve representing the slope r .

Lemma 6.1. *Let N be a compact 3-manifold whose boundary is a single torus, let $S \subset N$ be a closed connected incompressible surface, and let p be a prime. Then there exist infinitely many slopes r on ∂N for which the following conditions hold:*

- (1) *the inclusion homomorphism $H_1(N; \mathbb{Z}_p) \rightarrow H_1(N(r_i); \mathbb{Z}_p)$ is an isomorphism; and*
- (2) *S is incompressible in $N(r_i)$.*

Proof. There is a natural bijective correspondence between slopes on ∂M and unordered pairs of the form $\{c, -c\}$ where c is a primitive element of $L \doteq H_1(\partial N; \mathbb{Z})$. If r is a slope, the elements of the corresponding unordered pair are the homology classes defined by the two orientations of a simple closed curve representing c . If c is a primitive element of L we shall denote by r_c the slope corresponding to the pair $\{c, -c\}$.

Let $K \subset L$ denote the kernel of the natural homomorphism $H_1(\partial N; \mathbb{Z}_p) \rightarrow H_1(N; \mathbb{Z}_p)$. If c is a primitive class $c \in K$, it follows from the Mayer-Vietoris theorem that the inclusion homomorphism $H_1(N; \mathbb{Z}_p) \rightarrow H_1(N(r_c); \mathbb{Z}_p)$ is an isomorphism.

We fix a basis $\{\lambda, \mu\}$ of L such that $\lambda \in K$, and we identify L with an additive subgroup of the two-dimensional real vector space $V = H_1(\partial N; \mathbf{R})$. For each positive integer n , let $A_n \subset V$ denote the affine line $\mathbf{R}\lambda + n\mu \subset V$. Then $A_n \cap L \subset K$, and $A_n \cap L$ contains infinitely many primitive elements of L (for example the elements of the form $(knp+1)\lambda + n\mu$ for $k \in \mathbb{Z}$). In particular, K contains infinitely many primitive elements of L .

We distinguish two cases, depending on whether there (a) does or (b) does not exist an annulus in M having one boundary component in S and one in ∂M , and having interior disjoint from $S \cup \partial M$. In case (a) it follows from [5, Theorem 2.4.3] that there is a slope r_0 such that for every slope r whose geometric intersection number with r_0 is > 1 , the surface S is incompressible in $M(r)$. (For the application of [5, Theorem 2.4.3] we need to know that S is not boundary-parallel in M , but this is automatic since S has genus 2.) In particular there are three affine lines B_1, B_2 and B_3 in V such that for any primitive class $c \in L \setminus (B_1 \cup B_2 \cup B_3)$, the surface S is incompressible in $M(r_c)$. If we choose a natural number n large enough so that A_n is distinct from B_1, B_2 and B_3 , then $A_n \cap (B_1 \cup B_2 \cup B_3)$ consists of at most three points. Hence of the infinitely many primitive elements of L belonging to $A_n \cap L$, at most three lie in $B_1 \cup B_2 \cup B_3$. For any primitive element $c \in (A_n \cap L) \setminus (B_1 \cup B_2 \cup B_3)$, the slope r_c satisfies Conclusions (1) and (2).

In Case (b) it follows from [19, Theorem 1] that there are at most three slopes r for which S is compressible in $M(r)$. In particular, of the infinitely many elements $c \in K$ which are primitive in L , all but finitely many have the property that S is incompressible in $M(r_c)$. Hence there are infinitely many slopes c satisfying Conclusions (1) and (2). \square

Definition 6.2. If X is a compact orientable manifold with non-empty boundary then by the *double* of X we shall mean the quotient space DX obtained from $X \times \{0, 1\}$ by identifying $(x, 0)$ with $(x, 1)$ for each $x \in \partial X$. The involution of $X \times \{0, 1\}$ which interchanges $(x, 0)$ and $(x, 1)$ induces an orientation-reversing involution $\tau : DX \rightarrow DX$, which we shall call the *canonical involution* of DX . Define $\text{geodvol } X = \frac{1}{2}v_3\| [DX] \|$, where $\| [DX] \|$ denotes the Gromov norm of the fundamental class of DX , and v_3 is the volume of a regular ideal tetrahedron.

The following standard result does not seem to be in the literature.

Proposition 6.3. *Let X be a compact connected orientable 3-manifold with connected boundary S of genus greater than 1. Suppose that X is irreducible, boundary-irreducible and acylindrical. Then X admits a hyperbolic metric with totally geodesic boundary, and $\text{geodvol } X$ is the volume of this metric.*

Proof. The closed manifold DX is simple, and the surface S is incompressible in DX . Thus DX admits a complete hyperbolic metric by Thurston's Hyperbolization Theorem for Haken manifolds [15]. Let $\tau : DX \rightarrow DX$ be the canonical involution of DX . Fix a basepoint

$\star \in S$ and a basepoint $\tilde{\star}$ in the universal cover of DX which maps to \star . We identify DX with \mathbb{H}^3/Γ , where Γ is a Kleinian group. Using the basepoint $\tilde{\star}$, we identify $\pi_1(DX, \star)$ with Γ . Let $p : \mathbb{H}^3 \rightarrow DX$ be the covering projection, and let $\tilde{\tau} : \mathbb{H}^3 \rightarrow \mathbb{H}^3$ denote the lift of $\tau \circ p$ which fixes $\tilde{\star}$. Let \tilde{S} be the component of $p^{-1}(S)$ which contains $\tilde{\star}$. The map $\tilde{\tau}$ is then an orientation-reversing involution of \mathbb{H}^3 which fixes \tilde{S} .

Since τ is a homotopy equivalence, it follows from the proof of Mostow's rigidity theorem ([13], cf. [4]) that $\tilde{\tau}$ extends continuously to S_∞^2 , and that there is an isometry τ' of \mathbb{H}^3 whose extension to S_∞^2 agrees with that of $\tilde{\tau}$. In particular τ' is an orientation-reversing isometry of \mathbb{H}^3 whose restriction to S_∞^2 normalizes the restriction of Γ to S_∞^2 . Thus τ' normalizes Γ in the isometry group of \mathbb{H}^3 , and consequently τ' induces an involution of DX . The restriction of τ' to S_∞^2 also commutes with the restriction of each isometry in the image Δ of the inclusion homomorphism $\pi_1(S, \star) \rightarrow \Gamma$. It follows that τ' must be a reflection through a hyperbolic plane Π , where Π contains the axis of each element of Δ . In particular Π is invariant under Δ . Moreover, since the image of Π in DX is contained in the fixed set of an involution of DX , it must be a compact subsurface F .

Since F is covered by a hyperbolic plane, it is a totally geodesic surface in DX . Let $\Delta' \leq \Gamma$ denote the stabilizer of Π . Then the covering space $\widetilde{DX} = \mathbb{H}^3/\Delta'$ is homeomorphic to $F \times \mathbf{R}$. Let \tilde{F} denote the image of Π in \widetilde{DX} , so that \tilde{F} is the image of a lift of the inclusion of F into DX . Since $\Delta \leq \Delta'$, the inclusion of S into DX lifts to an embedding of S in \widetilde{DX} . Let \tilde{S} denote the image of this lift. The surfaces \tilde{S} and \tilde{F} are π_1 -injective and can be isotoped to a pair of disjoint surfaces which cobound a compact submanifold W of \widetilde{M} . According to [9, Theorem 10.5] W is homeomorphic to $\tilde{S} \times I$. This shows that \tilde{S} is isotopic to \tilde{F} in \widetilde{DX} , and hence that S is homotopic to F in DX . Since S is incompressible in DX , it follows from [18, Corollary 5.5] that S is isotopic to the totally geodesic surface F .

Since S and F are isotopic, there is an ambient isotopy of DX which carries X onto a submanifold of DX bounded by F . Pulling back the hyperbolic metric by the time-1 map of this isotopy endows X with the structure of a complete hyperbolic manifold with totally geodesic boundary. The isometry of DX induced by τ' fixes F and exchanges its complementary components. Hence, with the pulled back metric, X has half the hyperbolic volume of DX . Since the hyperbolic volume of DX is equal to $v_3\|DX\|$ (see [4, Theorem C.4.2]), this completes the proof. \square

The result below follows from a result of Agol-Storm-Thurston [2]

Proposition 6.4. *Let M be a closed, orientable hyperbolic 3-manifold containing a closed connected incompressible surface S such that $M \setminus S$ has an acylindrical component X . Then*

$$\text{vol } M \geq \text{geodvol } X.$$

Proof. Let $g : S \rightarrow M$ denote the inclusion map. Since S is a two-sided embedded surface, the family of all immersions of S in M which are homotopic to g is non-empty. Since M is \mathbb{P}^2 -irreducible, the main result of [8] asserts that this family contains a least area immersion $f : S \rightarrow M$, which is either an embedding or a 2-sheeted covering of a non-orientable surface.

Moreover, the second case arises only if S bounds a twisted I -bundle whose 0-section is isotopic to $f(S)$. Since f is locally area minimizing, $f(S)$ is a minimal surface.

It follows from [18, Corollary 5.5] that if f is an embedding then $f(S)$ is ambiently isotopic to S , and if f is a 2-sheeted covering map then $f(S)$ is isotopic to the 0-section of the twisted I -bundle bounded by S . Hence one of the components, say X , of $M \setminus f(S)$ is acylindrical. We may identify X with the path completion of a component X_0 of $M - f(S)$. Then the natural map $X \rightarrow M$ maps the interior of X homeomorphically onto X_0 and maps ∂X onto $f(S)$, either by a homeomorphism or by a 2-sheeted cover. The latter possibility arises exactly when S bounds a twisted I -bundle and $f(S)$ is a non-orientable surface. In particular, pulling back the hyperbolic metric on M under the natural map $X \rightarrow M$ gives X the structure of a complete hyperbolic manifold whose boundary is a minimal surface. Theorem 7.2 of [2] states that such a manifold X satisfies $\text{vol } X \geq \text{geodvol } X$.

Clearly we have $\text{vol } M \geq \text{vol } X$, so the proof is complete. \square

Theorem 6.5. *Let M be a closed, orientable hyperbolic 3-manifold containing a closed connected incompressible surface of genus 2, and suppose that $\text{Hg}(M) \geq 6$. Then M has volume greater than 6.45.*

Proof. According to Corollary 5.9, M contains a connected closed incompressible surface S of genus 2 such that either $M \setminus S$ has an acylindrical component, or $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$. If $M \setminus S$ has an acylindrical component X , by Proposition 6.3 X admits a hyperbolic metric with totally geodesic boundary, and the volume of X in this metric is equal to $\text{geodvol } X$. The main result of [12] asserts that this volume is greater than 6.45; hence by Proposition 6.4, $\text{vol } M$ is also greater than 6.45. On the other hand, if $\bar{\chi}(\text{kish}(M \setminus S)) \geq 2$, then Theorem 9.1 of [2] implies that M has volume greater than 7.32. \square

The following lemma is a strict improvement on Proposition 10.1 of [7]. The improvement is made possible by the results of [6].

Lemma 6.6. *Let M be a complete, finite-volume, orientable hyperbolic 3-manifold having exactly one cusp, such that $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 6$. Then either*

- (1) $\text{vol } M > 5.06$, or
- (2) M contains a genus-2 connected incompressible surface.

Proof. This is identical to the proof of [7, Proposition 10.1] except that

- each of the two appearances of the number 7 in the latter proof is replaced by 6, and
- the reference to the case $g = 2$ of [1, Theorem 8.13] is replaced by a reference to the case $g = 2$ of [6, Theorem 1.1].

\square

Theorem 6.7. *Let M be a complete, finite-volume, orientable hyperbolic 3-manifold having exactly one cusp, and suppose that*

$$\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 6.$$

Then M has volume greater than 5.06.

Proof. For a hyperbolic manifold satisfying the hypotheses of Theorem 6.7, Lemma 6.6 asserts that either M has volume greater than 5.06 or M contains a closed connected incompressible surface of genus 2. In the latter case, let N denote the compact core of M . According to Lemma 6.1 there is an infinite sequence of distinct slopes $(r_i)_{i \geq 1}$ on ∂N such that S is incompressible in each $N(r_i)$, and $\dim_{\mathbb{Z}_2} H_1(N(r_i); \mathbb{Z}_2) \geq 6$ for each i . The hyperbolic Dehn surgery theorem ([17], cf. [14]) asserts that $M_i \doteq N(r_i)$ is hyperbolic for all sufficiently large i , and hence after passing to a subsequence we may assume that all the M_i are hyperbolic. We now invoke Theorem 1A of [14], which implies that $\text{vol } M_i < \text{vol } M$ for all but finitely many i . (The authors of [14] attribute this particular consequence of their main result to Thurston.)

Now since $\dim_{\mathbb{Z}_2} H_1(M_i; \mathbb{Z}_2) \geq 6$ for each i , we have in particular that $\text{Hg}(M_i) \geq 6$ for each i . Since each M_i contains the genus-2 connected closed incompressible surface S , it now follows from Theorem 6.5 that $\text{vol } M_i > 6.45$ for each i . Hence $\text{vol } M > 6.45$. \square

Theorem 6.8. *Let M be a closed, orientable hyperbolic 3-manifold with*

$$\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 4$$

and suppose that the image of the cup product map $H^1(M; \mathbb{Z}_2) \otimes H^1(M; \mathbb{Z}_2) \rightarrow H^2(M; \mathbb{Z}_2)$ has dimension at most 1. Then M has volume greater than 3.08.

Proof. We recall that a group Γ is said to be k -free for a given positive integer k if every subgroup of Γ having rank at most k is free. According to Corollary 9.3 of [1], which was deduced from results in [3], if M is a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 3-free then $\text{vol } M > 3.08$.

Now suppose that M satisfies the hypotheses of Theorem 6.8, but that $\pi_1(M)$ is not 3-free. Fix a base point $P \in M$ and a subgroup X of $\pi_1(M, P)$ which has rank at most 3 and is not free. Let \bar{X} denote the image of X under the natural homomorphism $\eta : \pi_1(M, P) \rightarrow H_1(M; \mathbb{Z}_2)$. Then the subspace \bar{X} of $H_1(M; \mathbb{Z}_2)$ has dimension at most 3. Since $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 4$, there is a codimension-1 subspace V of $H_1(M; \mathbb{Z}_2)$ containing \bar{X} . Then $Y \doteq \eta^{-1}(V)$ is an index-2 subgroup of $\pi_1(M, P)$ containing X . Hence Y defines a 2-sheeted based covering space $p : (\tilde{M}, \tilde{P}) \rightarrow (M, P)$ such that $p_{\#} : \pi_1(\tilde{M}, \tilde{P}) \rightarrow \pi_1((M, P))$ maps some subgroup \tilde{X} of $\pi_1(\tilde{M}, \tilde{P})$ isomorphically onto X . In particular \tilde{X} has rank at most 3 and is not free, and so $\pi_1(\tilde{M})$ is not 3-free.

We now invoke Proposition 3.5 of [7], which asserts that if M is a closed, aspherical 3-manifold, if $r = \dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2)$, and if t denotes the dimension of the image of the cup product map $H^1(M; \mathbb{Z}_2) \otimes H^1(M; \mathbb{Z}_2) \rightarrow H^2(M; \mathbb{Z}_2)$, then for any integer $m \geq 0$ and any regular covering \tilde{M} of M with covering group $(\mathbb{Z}_2)^m$, we have $\dim_{\mathbb{Z}_2} H_1(\tilde{M}; \mathbb{Z}_2) \geq (m+1)r - m(m+1)/2 - t$. Taking M and \tilde{M} as above, the hypotheses of [7, Proposition 3.5] hold with $m = 1$, and by the hypothesis of the present theorem we have $r \geq 4$ and $t \leq 1$. Hence $\dim_{\mathbb{Z}_2} H_1(\tilde{M}; \mathbb{Z}_2) \geq 6$.

We next invoke Proposition 7.1 of [6], which implies that if $k \geq 3$ is an integer and if N is a closed simple 3-manifold such that $\dim_{\mathbb{Z}_2} H_1(N; \mathbb{Z}_2) \geq \max(3k - 4, 6)$, then either $\pi_1(N)$ is k -free, or N contains a closed connected incompressible surface of genus at most $k - 1$. We may apply this with $N = \widetilde{M}$ and $k = 3$, since we have seen that $\dim_{\mathbb{Z}_2} H_1(\widetilde{M}; \mathbb{Z}_2) \geq 6$. Since we have also seen that $\pi_1(\widetilde{M})$ is not 3-free, \widetilde{M} must contain a closed connected incompressible surface S of genus at most 2, and in view of simplicity, S must have genus exactly 2. Again using that $\dim_{\mathbb{Z}_2} H_1(\widetilde{M}; \mathbb{Z}_2) \geq 6$ —so that in particular $\text{Hg}(\widetilde{M}) \geq 6$ —we deduce from Theorem 6.5, with \widetilde{M} playing the role of M , that $\text{vol } \widetilde{M} > 6.45$. Hence

$$\text{vol } M = \frac{1}{2} \text{vol } \widetilde{M} > 3.225 > 3.08.$$

□

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DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE (M/C 249), UNIVERSITY OF ILLINOIS AT CHICAGO, 851 S. MORGAN ST., CHICAGO, IL 60607-7045

E-mail address: `culler@math.uic.edu`

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE (M/C 249), UNIVERSITY OF ILLINOIS AT CHICAGO, 851 S. MORGAN ST., CHICAGO, IL 60607-7045

E-mail address: `jdeblois@math.uic.edu`

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE (M/C 249), UNIVERSITY OF ILLINOIS AT CHICAGO, 851 S. MORGAN ST., CHICAGO, IL 60607-7045

E-mail address: `shalen@math.uic.edu`