Totally geodesic surfaces and homology

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Abstract

We construct examples of hyperbolic rational homology spheres and hyperbolic knot complements in rational homology spheres containing closed embedded totally geodesic surfaces.

1 Introduction

Let $M = \mathbb{H}^3/\Gamma$, $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ be an orientable hyperbolic 3-manifold and $f : F \to M$ a proper immersion of a connected, orientable surface of genus ≥ 2 such that $f_* : \pi_1(F) \to \Gamma$ is injective. F (or more precisely (f, F)) is said to be *totally geodesic* if $f_*(\pi_1(F)) \subset \Gamma$ is conjugate into $\mathrm{PSL}_2(\mathbb{R})$. Thurston and Bonahon have described the geometry of surface groups in hyperbolic 3-manifolds as falling into 3 classes: doubly degenerate groups, quasi-Fuchsian groups, and groups with accidental parabolics. The class of totally geodesic surface groups is a "positive codimension" subclass of the quasi-Fuchsian groups, so one may expect that hyperbolic 3-manifolds containing totally geodesic surface groups are special.

Indeed, the presence of a totally geodesic surface in a hyperbolic 3-manifold has important topological implications. Long showed [10] that immersed totally geodesic surfaces lift to embedded nonseparating surfaces in finite covers, proving the virtual Haken and virtually positive β_1 conjectures for hyperbolic manifolds containing totally geodesic surfaces. Given this, it is natural to wonder what topological constraints exist on hyperbolic 3-manifolds containing totally geodesic surfaces. Menasco-Reid have made the following conjecture [12]:

Conjecture (Menasco-Reid). *No hyperbolic knot complement in* S^3 *contains a closed embedded totally geodesic surface.*

They proved this conjecture for alternating knots. The "Menasco-Reid" conjecture has been shown true for many other classes of knots, including almost alternating knots [2], Montesinos knots [14], toroidally alternating knots [1], 3-bridge and double torus knots [7], and knots of braid index 3 [9] and 4 [11]. For a knot in one of the above families, any closed essential surface in its complement has a topological feature which obstructs it from being even quasi-Fuchsian. In general, however, one cannot hope to find such obstructions. Adams-Reid have given examples of closed embedded quasi-Fuchsian surfaces in knot complements which volume calculations prove not to be totally geodesic [3].

On the other hand, C. Leininger has given evidence for a counterexample by constructing a sequence of hyperbolic knot complements in S^3 containing closed embedded surfaces whose principal curvatures approach 0 [8]. In this paper, we take an alternate approach to giving evidence for a counterexample, proving

Theorem 1. *There exist infinitely many hyperbolic knot complements in rational homology spheres containing closed embedded totally geodesic surfaces.*

This answers a question of Reid—recorded as Question 6.2 in [8]—giving counterexamples to the natural generalization of the Menasco-Reid conjecture to knot complements in rational homology spheres. Thus the Menasco-Reid conjecture, if true, must reflect a deeper topological feature of knot complements in S^3 than simply their rational homology.

Prior to proving Theorem 1, in Section 2 we prove

Theorem 2. *There exist infinitely many hyperbolic rational homology spheres containing closed embedded totally geodesic surfaces.*

This seems of interest in its own right, and the proof introduces many of the techniques we use in the proof of Theorem 1. Briefly, we find a two cusped hyperbolic manifold containing an embedded totally geodesic surface which remains totally geodesic under certain orbifold surgeries on its boundary slopes, and use the Alexander polynomial to show that branched covers of these surgeries have no rational homology. In Section 3 we prove Theorem 1. In the final section, we give some idea of further directions and questions suggested by our approach.

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2 Theorem 2

Given a compact hyperbolic manifold M with totally geodesic boundary of genus g, gluing it to its mirror image $\bar{M}$ along the boundary yields a closed manifold DM—the "double" of M—in which the former ∂M becomes an embedded totally geodesic surface. One limitation of this construction is that this surface contributes half of its first homology to the first homology of DM, so that $\beta_1(DM) \geq g$. This is well-known, but we include an argument to motivate our approach. Consider the relevant portion of the rational homology Mayer-Vietoris sequence for DM:

$$\cdots \to \mathrm{H}_1(\partial M, \mathbb{Q}) \stackrel{(i_*, -j_*)}{\longrightarrow} \mathrm{H}_1(M, \mathbb{Q}) \oplus \mathrm{H}_1(\bar{M}, \mathbb{Q}) \to \mathrm{H}_1(DM, \mathbb{Q}) \to 0$$

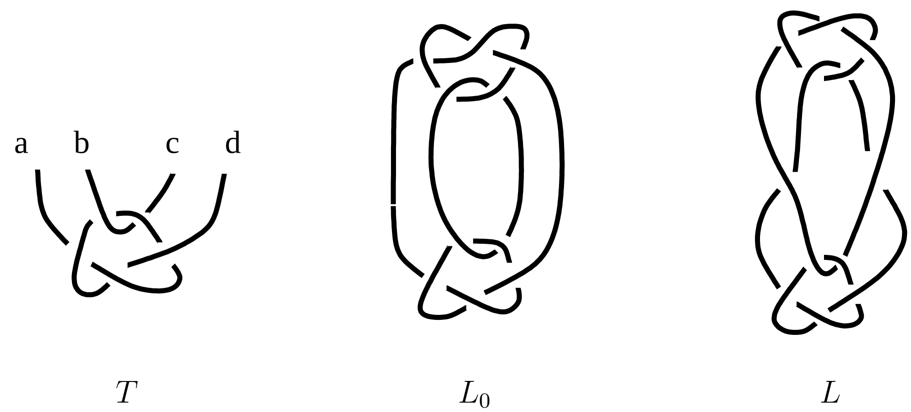


Figure 1: The tangle T and its double and twisted double.

The labeled maps i_* and j_* are the maps induced by inclusion of the surface into M and $\bar{M}$, respectively. Recall that by the "half lives, half dies" lemma (see eg. [5], Lemma 3.5), the dimension of the kernel of i_* is equal to g. Hence $\beta_1(M) \geq g$. Since the gluing isometry $\partial M \to \partial \bar{M}$ (the identity) extends over M, ker $i =$ ker j, and so dim $\mathrm{Im}(i_*, -j_*) = g$. Hence

$$\mathrm{H}_1(DM) \cong \frac{\mathrm{H}_1(M) \oplus \mathrm{H}_1(\bar{M})}{\mathrm{Im}((i_*, -j_*))}$$

has dimension at least g.

Considering the above picture gives hope that by cutting DM along ∂M and regluing via some isometry $\phi : \partial M \to \partial M$ to produce a "twisted double" $D_\phi M$, one may reduce the homological contribution of ∂M. For then $j = i \circ \phi$, and if ϕ moves the kernel of the inclusion off of itself, then the argument above shows that the homology of $D_\phi M$ will be reduced. Below we apply this idea to a family of examples constructed by Zimmerman and Paoluzzi [20] which build on the "Tripus" example of Thurston [18].

The complement in the ball of the tangle T in Figure 1 is one of the minimal volume hyperbolic manifolds with totally geodesic boundary, obtained as an identification space of a regular ideal octahedron [13]. We will denote it O_∞. For $n \geq 3$, the orbifold O_n with totally geodesic boundary, consisting of the ball with cone locus T of cone angle $2\pi/n$, has been explicitly described by Zimmerman and Paoluzzi [20] as an identification space of a truncated tetrahedron. For each $k < n$ with $(k, n) = 1$, Zimmerman and Paoluzzi describe a hyperbolic manifold $M_{n,k}$ which is an n-fold branched cover of O_n. Topologically, $M_{n,k}$ is the n-fold branched cover of the ball, branched over T, obtained as the kernel of $\langle x, y\rangle = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} = \langle t\rangle$ via $x \mapsto t$, $y \mapsto t^k$, where x and y are homology classes representing meridians of the two components of T.

We recall a well-known fact about isometries of spheres with 4 cone points:

Fact. *Let S be a hyperbolic sphere with 4 cone points of equal cone angle α, $0 \leq \alpha \leq 2\pi/3$, labeled a, b, c, d. Then there is an orientation-preserving isometry realizing each of the following permutations of the cone points:*

$$(ab)(cd) \qquad (ac)(bd) \qquad (ad)(bc)$$

Using this fact, and abusing notation, let ϕ be the isometry $(ab)(cd)$ of ∂O_n, with labels as in Figure 1. Doubling the tangle ball produces the link L_0 in Figure 1, and cutting along the separating 4-punctured sphere and regluing via ϕ produces the link L, a *mutant* of L_0 in the classical terminology. Note that L and all of the orbifolds $D_\phi O_n$ contain the mutation sphere as a totally geodesic surface, by the fact above. ϕ lifts to an isometry $\tilde{\phi}$ of $\partial M_{n,k}$, and the twisted double $D_{\tilde{\phi}} M_{n,k}$ is the corresponding branched cover over L.

The homology of $D_{\tilde{\phi}} M_{n,k}$ can be described using the Alexander polynomial of L. The two-variable Alexander polynomial of L is

$$\Delta_L(x,y) = \frac{1}{x^3}(x-1)(xy-1)(y-1)^2(x-y)$$

For the regular $\mathbb{Z}$-covering of $S^3 - L$ given by $x \mapsto t^k$, $y \mapsto t$, the Alexander polynomial is

$$\Delta_L^k(t) = (t-1)\Delta(t^k,t) = \frac{1}{t^{3k-1}}(t-1)^5 \nu_{k-1}(t)\nu_k(t)\nu_{k+1}(t)$$

where $\nu_k(t) = t^{k-1} + t^{k-2} + \cdots + t + 1$. By a theorem originally due to Sumners [17] in the case of links, the first Betti number of $D_{\tilde{\phi}} M_{n,k}$ is the number of roots shared by $\Delta_k(t)$ and $\nu_n(t)$. Since this number is 0 for many n and k, we have a more precise version of Theorem 1. For example, we have

Theorem. *For $n > 3$ prime and $k \neq 0, 1, n-1$, $D_{\tilde{\phi}} M_{n,k}$ is a rational homology sphere containing an embedded totally geodesic surface.*

The techniques used above are obviously more generally applicable. Given any hyperbolic two-string tangle in a ball with totally geodesic boundary, one may double it to get a 2-component hyperbolic link in $\mathbb{S}^3$ and then mutate along the separating 4-punctured sphere by an isometry. By the hyperbolic Dehn surgery theorem and the fact above, for large enough n, $(n, 0)$ orbifold surgery on each component will yield a hyperbolic orbifold with a separating totally geodesic orbisurface. Then n-fold manifold branched covers can be constructed as above. One general observation about such covers follows from the following well-known fact, originally due to Conway:

Fact. *The one-variable Alexander polynomial of a link is not altered by mutation; ie,*

$$\Delta_{L_0}(t,t) = \Delta_L(t,t)$$

when L is obtained from L_0 by mutation along a 4-punctured sphere.

In our situation, this implies the following:

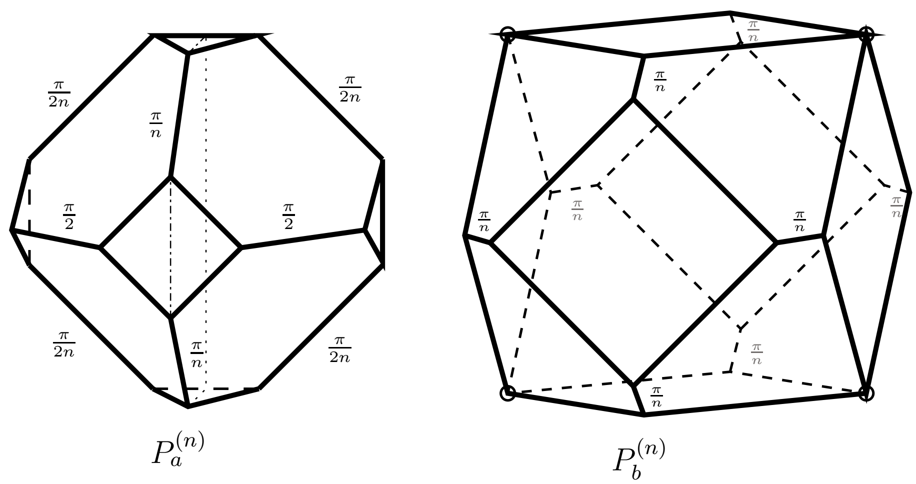


Figure 2: Cells for O_n

Corollary. *A 2-component link in* S^3 *which is the twisted double of a tangle has no integral homology spheres among its abelian branched covers.*

Proof. A link L_0 which is the double of a tangle has Alexander polynomial 0. Therefore by the fact above,

$$\Delta_L^1(t) = (t-1)\Delta_L(t,t) = (t-1)\Delta_{L_0}(t,t) = 0$$

and so $D_{\tilde{\phi}}M_{n,1}$ has positive first Betti number by Sumners' theorem. The canonical abelian n^2-fold branched cover of L covers $D_{\tilde{\phi}}M_{n,1}$ and so also has positive first Betti number. Since the other n-fold branched covers of L have n-torsion, no branched covers of L have trivial first homology.

□

3 Theorem 1

In this section we construct hyperbolic knot complements in rational homology spheres containing closed embedded totally geodesic surfaces. This is accomplished by producing a 3-cusped manifold N containing an embedded totally geodesic 4-punctured sphere which intersects two of the cusps, and using a polyhedral decomposition to show that the 4-punctured sphere remains totally geodesic under n-fold orbifold surgery on each of its boundary slopes, for $n \geq 3$. We then adapt an argument of Sakuma [16] to show that certain one-cusped manifold covers M_n of the resulting orbifolds O_n have a Dehn filling yielding a rational homology sphere.

We first discuss the orbifolds O_n mentioned above. For each n, O_n decomposes into the two polyhedra in Figure 2. Realized as a hyperbolic polyhedron,

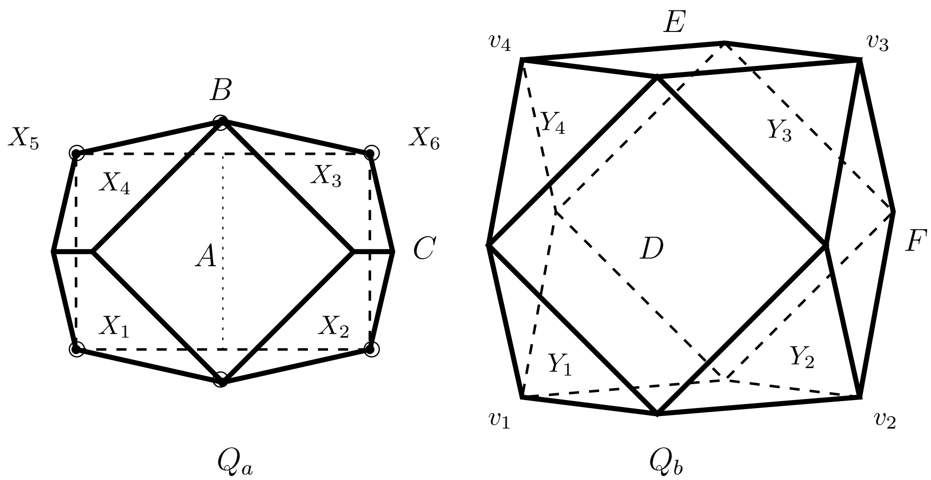


Figure 3: Cells for M_0

$P_a^{(n)}$ is composed of two truncated tetrahedra, each of which has two opposite edges of dihedral angle $\pi/2$ and all other dihedral angles $\pi/2n$, glued along a face. This decomposition is indicated in Figure 3 by the lighter dashed and dotted lines. The polyhedron $P_b^{(n)}$ has all edges with dihedral angle $\pi/2$ except for those labeled otherwise, and realized as a hyperbolic polyhedron it has all visible symmetries and all circled vertices at infinity. By Andreev's theorem, polyhedra with the desired properties exist in hyperbolic space. Certain face pairings (described below) of $P_a^{(n)}$ yield a compact hyperbolic orbifold with totally geodesic boundary a sphere with 4 cone points of cone angle $2\pi/n$, and faces of $P_b^{(n)}$ can be glued to give a one-cusped hyperbolic orbifold with a torus cusp and totally geodesic boundary a sphere with 4 cone points of cone angle $2\pi/n$. These boundary spheres are isometric, and O_n is formed by gluing the faces together.

The geometric limit of the O_n as $n \to \infty$ is N, a 3-cusped manifold which decomposes into the two polyhedra in Figure 3. As above, realized as a convex polyhedron in hyperbolic space Q_a has all circled vertices at infinity. The edge of Q_a connecting face A to face C is finite length, as is the corresponding edge on the opposite vertex of A; all others are ideal or half-ideal, and all have dihedral angle $\pi/2$. Q_a has a reflective involution of order 2 corresponding to the involution of $P_a^{(n)}$ interchanging the two truncated tetrahedra. The fixed set of this involution on the back face is shown as a dotted line, and notationally we regard Q_a as having an edge there with dihedral angle π, splitting the back face into two faces X_5 and X_6. Q_b is the regular all-right hyperbolic ideal cuboctahedron.

Another remark on notation: the face opposite a face labeled with only

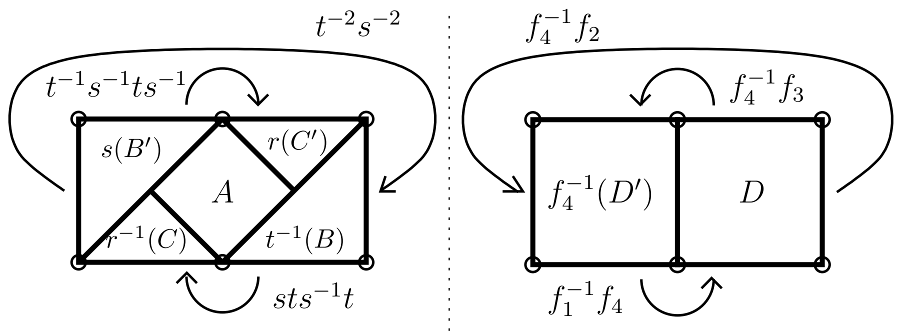


Figure 4: Totally Geodesic Faces of N_a and N_b

a letter should be interpreted as being labeled with that letter "prime". For instance, the leftmost triangular face of Q_a has label C'. Also, each "back" triangular face of Q_b takes the label of the face with which it shares a vertex. For example, the lower left back triangular face is Y_1'.

We first consider face pairings of Q_a producing a manifold N_a with two annulus cusps and totally geodesic boundary. Let r, s, and t be isometries realizing face pairings $X_1 \mapsto X_3$, $X_6 \mapsto X_4$, and $X_2 \mapsto X_5$, respectively. Poincaré's polyhedron theorem yields a presentation

$$\langle\, r, s, t \mid rst = 1 \,\rangle$$

for the group generated by r, s, and t. Note that this group is free on two generators, say s and t, where by the relation $r = t^{-1}s^{-1}$. Choose as the "boundary subgroup" (among all possible conjugates) the subgroup fixing the hyperbolic plane through the face A. A fundamental polyhedron for this group and its face-pairing isometries are in Figure 4. Note that the boundary is a 4-punctured sphere, and two of the three generators listed are the parabolics $t^{-1}s^{-1}ts^{-1}$ and $sts^{-1}t$, which generate the two annulus cusp subgroups of $\langle\, s, t \,\rangle$.

We now turn our attention to Q_b and the 3-cusped quotient manifold N_b. For $i \in \{1, 2, 3, 4\}$, let f_i be the isometry pairing the face $Y_i \to Y'_{i+1}$ so that $v_i \mapsto v_{i+1}$. Let g_1 be the *hyperbolic* isometry (that is, without twisting) sending $E \to E'$, and g_2 the hyperbolic isometry sending $F \to F'$. The polyhedron theorem gives presentation

$$\begin{aligned}\langle f_1, f_2, f_3, f_4, g_1, g_2 \quad | \quad & f_1 g_2 f_2^{-1} g_1^{-1} = 1, \\ & f_2^{-1} g_2^{-1} f_3 g_1^{-1} = 1, \\ & f_3 g_2^{-1} f_4^{-1} g_1 = 1, \\ & f_4^{-1} g_2 f_1 g_1 = 1 \rangle\end{aligned}$$

for the group generated by the face pairings. The first 3 generators and relations may be eliminated from this presentation using Nielsen-Schreier transfor-

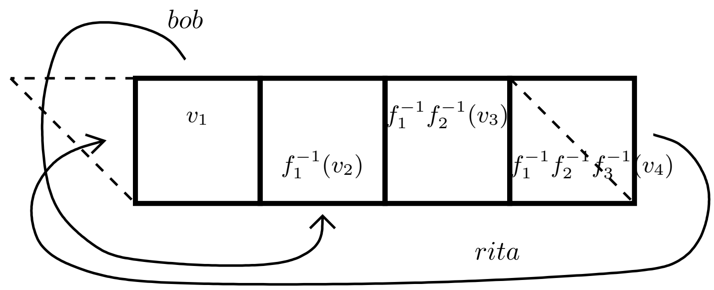


Figure 5: Closed Cusp of M_b

mations, yielding a presentation

$$\langle\ f_4, g_1, g_2 \mid f_4^{-1}[g_2, g_1]f_4[g_2, g_1^{-1}] = 1\ \rangle$$

(our commutator convention is $[x, y] = xyx^{-1}y^{-1}$), where the first 3 relations yield

$$f_1 = g_1g_2^{-1}g_1^{-1}f_4g_2g_1^{-1}g_2^{-1},\ f_2 = g_2^{-1}g_1^{-1}f_4g_2g_1^{-1},\ f_3 = g_1^{-1}f_4g_2$$

The second presentation makes clear that the homology of N_b is free of rank 3, since each generator has exponent sum 0 in the relation. Faces D and D' make up the totally geodesic boundary of N_b. In Figure 4 is a fundamental polyhedron for the boundary subgroup fixing D, together with the face pairings generating the boundary subgroup.

N_b has two annulus cusps, each with two boundary components on the totally geodesic boundary, and one torus cusp. A fundamental domain for the torus cusp in the horosphere centered at v_1 is shown in Figure 5, together with face pairing isometries generating the rank 2 parabolic subgroup fixing v_1. The generators shown are

$$bob = (f_4g_1^{-1})^2f_4g_2g_1^{-1}g_2^{-1},\ rita = (f_4g_1^{-1})^3f_4g_2g_1^{-1}g_2^{-1}$$

Note that $(bob)^{-4}(rita)^3$ is zero in homology. This and $rita \cdot (bob)^{-1} = f_4g_1^{-1}$ together generate the cusp subgroup fixing v_1. For later convenience, we now switch to the conjugate of this subgroup by f_4^{-1}, fixing v_4, and refer to $m = f_4^{-1}(f_4g_1^{-1})f_4 = g_1^{-1}f_4$ and $l = f_4^{-1}((bob)^{-4}(rita)^3)f_4$ as a "meridian-longitude" generating set for the closed cusp of N_b.

The totally geodesic 4-punctured spheres on the boundaries of N_a and N_b are each the double of a regular ideal rectangle, and we construct N by gluing N_a to N_b along them. Let us therefore assume that the polyhedra in Figure 4 are realized in hyperbolic space in such a way that face A of Q_a and face D of Q_b are in the same hyperbolic plane, with Q_a and Q_b in opposite half-spaces. Further arrange so that the polyhedra are aligned in the way suggested by folding the page containing Figure 4 along the dotted line down the center of the figure.

With this arrangment, Maskit's combination theorem gives a presentation for the amalgamated group:

$$\begin{aligned}\langle\ f_4, g_1, g_2, s, t \quad | \quad & f_4^{-1}[g_2, g_1] f_4 [g_2, g_1^{-1}] = 1,\\ & t^{-2}s^{-2} = f_4^{-1} g_2^{-1} g_1^{-1} f_4 g_2 g_1^{-1},\\ & sts^{-1}t = g_2 g_1 g_2^{-1} f_4^{-1} g_1 g_2 g_1^{-1} f_4,\\ & t^{-1}s^{-1}ts^{-1} = f_4^{-1} g_1^{-1} f_4 g_2\ \rangle\end{aligned}$$

The first relation comes from N_b and the others come from setting the boundary face pairings equal to each other. Observe that the last relation can be solved for g_2. Using Nielsen-Schreier transformations to eliminate g_2 and the last relation, the resulting presentation is

$$\begin{aligned}&\langle\ f_4, g_1, s, t \mid\\ &\quad f_4^{-2} g_1 [f_4 t^{-1}s^{-1}ts^{-1}, g_1] f_4 [f_4 t^{-1}s^{-1}ts^{-1}, g_1^{-1}] g_1^{-2} f_4 g_1 = 1\\ &\quad t^{-2}s^{-2} = f_4^{-1} st^{-1}st f_4^{-1} g_1^{-1} f_4^2 t^{-1}s^{-1}ts^{-1} g_1^{-1},\\ &\quad sts^{-1}t = f_4^{-1} g_1 f_4 t^{-1}s^{-1}ts^{-1} g_1 st^{-1}st f_4^{-2} g_1 f_4 t^{-1}s^{-1}ts^{-1} g_1^{-1} f_4\ \rangle\end{aligned}$$

Replace g_1 with the meridian generator of the closed cusp of N_b, $m = g_1^{-1} f_4$, and add the two conjugate generators $m_1 = f_4^{-1} m f_4$ and $m_2 = st^{-1}stm_1t^{-1}s^{-1}ts^{-1}$ to get

$$\begin{aligned}&\langle\ f_4, m, m_1, m_2, s, t \mid\\ &\quad m_1 = f_4^{-1} m f_4,\ m_2 = st^{-1}stm_1t^{-1}s^{-1}ts^{-1}\\ &\quad m_1^{-1}t^{-1}s^{-1}ts^{-1}f_4m^{-1}m_2mf_4^{-1}st^{-1}stm_1m^{-1} = 1\\ &\quad s^2t^2f_4^{-1}m_2mf_4^{-1} = 1\\ &\quad t^{-1}st^{-1}s^{-1}m^{-1}f_4t^{-1}s^{-1}ts^{-1}f_4m^{-1}m_2^{-1}m = 1\ \rangle\end{aligned}$$

Note that after abelianizing, each of the last two relations expresses $f_4^2 = m^2s^2t^2$, since m_1 and m_2 are conjugate to m and therefore identical in homology. In light of this, we replace f_4 by $u = t^{-1}s^{-1}f_4m^{-1}$, which has order 2 in homology. This yields

$$\begin{aligned}&\langle\ m, m_1, m_2, s, t, u \mid\\ &\quad m_1^{-1}m^{-1}u^{-1}t^{-1}s^{-1}mstum = 1 && (1)\\ &\quad m_2^{-1}st^{-1}stm_1t^{-1}s^{-1}ts^{-1} = 1 && (2)\\ &\quad m_1^{-1}t^{-1}s^{-1}t^2um_2u^{-1}t^{-2}stm_1m^{-1} = 1 && (3)\\ &\quad s^2t^2m^{-1}u^{-1}t^{-1}s^{-1}m_2u^{-1}t^{-1}s^{-1} = 1 && (4)\\ &\quad t^{-1}st^{-1}s^{-1}m^{-1}stumt^{-1}s^{-1}t^2um_2^{-1}m = 1\ \rangle && (5)\end{aligned}$$

Let R_i denote the relation labeled (i) in the presentation above. In the abelianization, R_1 and R_2 set $m_1 = m$ and $m_2 = m_1$, respectively, R_3 disappears, and the last two relations set $u^2 = 1$. Therefore

$$\mathrm{H}_1(N) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z} = \langle m\rangle \oplus \langle s\rangle \oplus \langle t\rangle \oplus \langle u\rangle$$

(In this paper we will generally blur the distinction between elements of π_1 and their homology classes.)

The boundary slopes of the totally geodesic 4-punctured sphere coming from ∂N_a and ∂N_b are represented in $\pi_1(N)$ by $t^{-1}s^{-1}ts^{-1}$ and $sts^{-1}t$. Let O_n be the finite volume hyperbolic orbifold produced by performing face identifications on $P_a^{(n)}$ and $P_b^{(n)}$ corresponding to those on Q_a and Q_b. O_n is geometrically produced by n-fold orbifold filling on each of the above boundary slopes of N. Appealing to the polyhedral decomposition, we see that the separating 4-punctured sphere remains totally geodesic, becoming a sphere with 4 cone points of order n. Our knots in rational homology spheres are certain manifold covers of the O_n. In order to understand the homology of these manifold covers, we compute the homology of the corresponding abelian covers of N.

Let $p : \widetilde{N} \to N$ be the maximal free abelian cover; that is, $\widetilde{N}$ is the cover corresponding to the kernel of the map $\pi_1(N) \to \mathrm{H}_1(N) \to \mathbb{Z}^3 = \langle x, y, z \rangle$ by

$$m \mapsto x \qquad s \mapsto y \qquad t \mapsto z \qquad u \mapsto 1$$

Let X be a standard presentation 2-complex for $\pi_1(N)$ and $\widetilde{X}$ the 2-complex covering X corresponding to $\widetilde{N} \to N$. Then the first homology and Alexander module of $\widetilde{X}$ are naturally isomorphic to those of $\widetilde{N}$, since N is homotopy equivalent to a cell complex obtained from X by adding cells of dimension 3 and above. The covering group $\mathbb{Z}^3$ acts freely on the chain complex of $\widetilde{X}$, so that it is a free $\mathbb{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$-module. Below we give a presentation matrix for the Alexander module of $\widetilde{X}$:

$$\begin{pmatrix} \frac{1-yz+xyz}{x^2yz} & 0 & -1 & -\frac{y^2z^2}{x} & \frac{-1+yz+z^2}{xz^2} \\ -\frac{1}{x} & \frac{y^2}{x} & \frac{x-1}{x} & 0 & 0 \\ 0 & -\frac{1}{x} & \frac{z}{xy} & \frac{yz}{x} & -\frac{1}{x} \\ \frac{x-1}{x^2yz} & -\frac{(x-1)(y+z)}{xz} & \frac{x-1}{xyz} & \frac{y(x-z)}{x} & \frac{1-2x+xz}{xz^2} \\ \frac{x-1}{x^2z} & -\frac{y(x-1)(y-1)}{xz} & \frac{(x-1)(-1+y-z)}{xyz} & \frac{y(-x+xy+xyz-yz)}{x} & \frac{x+y-2xy}{xz^2} \\ \frac{x-1}{x^2} & 0 & -\frac{z(x-1)}{xy} & -\frac{yz(x+yz)}{x} & \frac{y+xz}{xz} \end{pmatrix}$$

The rows of the matrix above correspond to lifts of the generators for $\pi_1(N)$ sharing a basepoint, ordered as $\{\tilde{m}, \tilde{m_1}, \tilde{m_2}, \tilde{s}, \tilde{t}, \tilde{u}\}$ reading from the top down. These generate $\mathrm{C}_1(\widetilde{X})$ as a $\mathbb{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$-module. The columns are the Fox free derivatives of the relations in terms of the generators, giving a basis for the image of $\partial \mathrm{C}_2(\widetilde{X})$. For a generator g above, let p_g be the determinant of the square matrix obtained by deleting the row corresponding to $\tilde{g}$. These polynomials are

$$\begin{aligned} p_m &= \tfrac{-1}{x^4z^2}(x-1)^2(y-1)(z-1)(y+z+4yz+y^2z+yz^2) \\ p_{m_1} &= \tfrac{1}{x^4z^2}(x-1)^2(y-1)(z-1)(y+z+4yz+y^2z+yz^2) \\ p_{m_2} &= \tfrac{-1}{x^4z^2}(x-1)^2(y-1)(z-1)(y+z+4yz+y^2z+yz^2) \\ p_s &= \tfrac{1}{x^4z^2}(x-1)(y-1)^2(z-1)(y+z+4yz+y^2z+yz^2) \\ p_t &= \tfrac{-1}{x^4z^2}(x-1)(y-1)(z-1)^2(y+z+4yz+y^2z+yz^2) \\ p_u &= 0 \end{aligned}$$

The Alexander polynomial of $H_1(\widetilde{N})$ is the greatest common factor:

$$\Delta(x, y, z) = (x-1)(y-1)(z-1)(y + z + 4yz + y^2z + yz^2)$$

up to multiplication by an invertible element of $\mathbb{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$.

Let N_∞ be the infinite cyclic cover of N factoring through $\widetilde{N}$ given by

$$m \mapsto x^2 \qquad s \mapsto x \qquad t \mapsto x \qquad u \mapsto 1$$

Then the chain complex of N_∞ is a Λ-module, where $\Lambda = \mathbb{Z}[x, x^{-1}]$, and specializing the above picture yields an Alexander polynomial

$$\begin{aligned} \Delta_\infty(x) &= (x^2-1)(x-1)^2(2x + 4x^2 + 2x^3) \\ &= 2x(x-1)^3(x+1)^3 \end{aligned}$$

Let N_n be the n-fold cyclic cover of N factoring through N_∞. In the case that n is odd, we adapt an argument of Sakuma [16] to show that the "branched cover" given by filling N_n along the lifts of the slopes corresponding to m, $sts^{-1}t$, and $t^{-1}s^{-1}ts^{-1}$ is a rational homology sphere. For n odd, N_n has three cusps, since m, $sts^{-1}t$, and $t^{-1}s^{-1}ts^{-1}$ map to $x^{\pm 2}$, which generates $\mathbb{Z}/n\mathbb{Z}$.

The chain complex of N_n is isomorphic to $C_*(N_\infty) \otimes (\Lambda/(x^n - 1))$. The short exact sequence of coefficient modules

$$0 \to \mathbb{Z} \xrightarrow{2\nu_n} \Lambda/(x^n - 1) \to \Lambda/(2\nu_n) \to 0$$

where the map on the left is multiplication by $2\nu_n$, gives rise to a short exact sequence in homology

$$0 \to H_1(N) \xrightarrow{2tr} H_1(N_n) \to H_1(N_\infty)/2\nu_n H_1(N_\infty) \to 0$$

where tr is the transfer map, $tr(h) = h + x.h + \cdots + x^{n-1}.h$ for a homology class h. Define $H_n = H_1(N_\infty)/2\nu_n H_1(N_\infty)$. Since the Alexander polynomial of N_∞ does not share roots with $2\nu_n$, H_n is a torsion $\mathbb{Z}$-module.

In the setting that N is a link complement in S^3, Sakuma observes that it follows immediately from the fact that fundamental groups of link complements are meridionally generated that H_n (using ν_n instead of $2\nu_n$) is the first homology group of the cover of S^3 branched over the link. In our setting N is not a link complement in S^3—it has torsion in H_1—and the result is not immediate.

Let S_n be the closed 3-manifold obtained by filling N_n along the lifts of the slopes corresponding to $sts^{-1}t$, $t^{-1}s^{-1}ts^{-1}$, and m. We record the desired result as a lemma:

Lemma. *For $n \geq 3$, S_n is a rational homology sphere.*

Proof. As suggested above, we relate the first homology of S_n to the torsion group H_n. Note that in $H_1(N)$, the specified slopes represent $2t$, $-2s$, and m. The kernel of the map $H_1(N_n) \to H_n$ is generated by twice the transfers of the generators of $H_1(N)$. The Mayer-Vietoris sequence implies that $H_1(S_n)$ is the quotient of $H_1(N_n)$ by the subgroup generated by $tr(2t)$, $tr(-2s)$, and $tr(m)$. Hence if $2tr(u)$ is also contained in the kernel of the map to $H_1(S_n)$, it follows that $H_1(S_n)$ is a further quotient of H_n and therefore torsion.

Observe that in the chain complex of N_∞, the free derivative of R_4 is

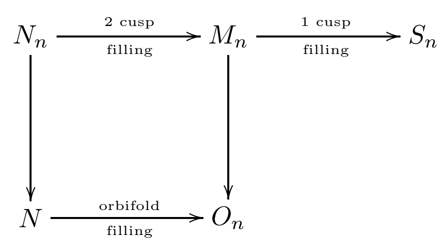
$$\partial\tilde{R}_4 = -x^2\tilde{m} + \tilde{m}_2 + (x-1)\tilde{s} + (x^3 + x^2 - 2x)\tilde{t} - 2x^2\tilde{u}$$

In the chain complex of N_n, after summing all translates of the above by powers of x the $\tilde{s}$ and $\tilde{t}$ terms telescope, leaving

$$\partial tr(R_4) = -tr(m) + tr(m_2) - 2tr(u)$$

Since m and m_2 are homologous, $2tr(u)$ is trivial in $H_1(N_n)$. The Lemma follows. $\square$

Let M_n be the manifold obtained by filling two of the three cusps of N_n along the slopes covering $sts^{-1}t$ and $t^{-1}s^{-1}ts^{-1}$. For the convenience of the reader, we now offer the following "commutative diagram" relating the various coverings and fillings of N:



M_n is a branched cover of O_n, which we have geometrically described as produced by n-fold orbifold filling along $sts^{-1}t$ and $t^{-1}s^{-1}ts^{-1}$. M_n contains a closed totally geodesic surface covering the totally geodesic sphere with 4 cone points in O_n. S_n is produced by filling the remaining cusp of M_n along the meridian covering m to give a closed manifold. Since S_n is a rational homology sphere, M_n is a knot complement in a rational homology sphere, and we have proven Theorem 1.

4 Further directions

Performing ordinary Dehn filling along the 3 meridians of N specified in the previous section yields a manifold S, which is easily seen to be the connected sum of two spherical manifolds. The half arising from the truncated tetrahedra is the quotient of S^3, regarded as the set of unit quaternions, by the subgroup $\langle i, j, k\rangle$. The half arising from the cuboctahedron is the lens space $L(4, 1)$. Thus the manifolds S_n may be regarded as n-fold branched covers over a 3-component link in S.

This fact suggests that a similar approach to that of the section above may be used to produce closed embedded totally geodesic surfaces in knot complements in S^3. For example, Ushijima has shown that for each $n \geq 3$ a certain n-fold branched cover of the tangle T from Section 1 is a hyperbolic manifold with totally geodesic boundary which embeds in S^3 as the complement of a knotted handlebody of genus $n-1$ [19]. (For $n = 3$, Zimmerman-Paoluzzi showed in [20] that this is the Tripus). A genus $n-1$ handlebody may be obtained as the n-fold branched cover of a ball over the trivial 2-string tangle, so knot complements in the genus $n-1$ handlebody may be obtained as n-fold branched covers over the trivial tangle of a knot complement in the ball. In analogy with Section 3, allowing the complement of T to play the role of N_a we ask

Question. *Does there exist a hyperbolic* 3*-manifold with one rank 2 and two rank 1 cusps and totally geodesic boundary isometric to the totally geodesic boundary of the complement of the tangle T, which is the complement of a tangle in the ball?*

Such a manifold would furnish an analog of the manifold N_b in Section 3, making the glued manifold N a 3-component link complement in S^3 containing a totally geodesic Conway sphere intersecting two of the components. If this sphere remained totally geodesic under the right orbifold surgery along its boundary slopes, branched covers would give a counterexample to the Menasco-Reid conjecture. In any case, Thurston's hyperbolic Dehn surgery theorem implies that as $n \to \infty$, the resulting surfaces would have principal curvature approaching 0, furnishing new examples of the phenomenon discovered by Leininger in [8] (although unlike Leininger's examples this would not give bounded genus).

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