TRIGONOMETRY OF PARTIALLY TRUNCATED HYPERBOLIC TRIANGLES AND TETRAHEDRA

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For $n = 4$, 5 or 6, an n-gon can be regarded as a *partially truncated* triangle, resulting from removing open affine neighborhoods of certain vertices of a triangle. This has particular significance in hyperbolic geometry when the edges resulting from truncation are at right angles to the others. Multiple versions of the hyperbolic laws of sines and cosines are proved for such polygons in Chapter VI of Fenchel's *Elementary geometry in hyperbolic space* [3].

The first results of this note, proved in Section 2, establish additional such laws which are not already in [3] nor in Ratcliffe's *Foundations of hyperbolic manifolds* [4]. Proposition 2.1 pertains to hyperbolic quadrilaterals, and 2.2 to pentagons, in which the non-truncated vertices are *ideal* (see Section 1 below). In these results, the length of a horospherical cross-section at an ideal vertex plays the role that would be played by the dihedral angle at an ordinary vertex.

In Section 3 we turn to the analogous situation in dimension three. Our second collection of results concern *partially truncated tetrahedra* in $\mathbb{H}^3$, which are homeomorphic to affine 3-simplices with certain vertices or their open neighborhoods removed. Proposition 3.5 bounds the *transversal length* of such a tetrahedron, meaning the minimum distance between a specified pair of opposite internal edges, in terms of the tetrahedron's set of internal edge lengths. Here an edge is *internal* if it is not contained in a face resulting from truncation.

In both settings we follow the approach of [4, Ch. 3] and use the *hyperboloid model* for $\mathbb{H}^n$, where it is taken as a subset of $\mathbb{R}^{n+1}$ equipped with the *Lorentzian inner product*, a certain non positive-definite bilinear form. Vectors of the ambient $\mathbb{R}^{n+1}$ carry information about different objects of $\mathbb{H}^n$, depending on the sign of their self-pairing. We introduce this perspective in Section 1 below. One can leverage it to encode partially truncated triangles using just three vectors, and partially truncated tetrahedra using four, as we subsequently do.

The results of this note are applied to help bound volumes of hyperbolic 3-manifolds with totally geodesic boundary below, in other papers that have appeared or will soon.

1. Background: the meaning of vectors in the hyperboloid model

We begin by reviewing Ratcliffe's notation from Chapter 3 of [4], which we will generally follow in describing the hyperboloid model of hyperbolic space. The *Lorentzian inner product* of $\mathbf{x} = (x_1, \ldots, x_{n+1})$ and $\mathbf{y} = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$ is defined as

$$\mathbf{x} \circ \mathbf{y} = -x_1y_1 + x_2y_2 + \ldots + x_{n+1}y_{n+1},$$

and $\mathbf{x}$ is said to be *space-like*, *light-like*, or *time-like* respectively as $\mathbf{x} \circ \mathbf{x}$ is positive, zero, or negative. The *Lorentzian norm* of $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\mathbf{x} \circ \mathbf{x}}$, where the square root is taken to be positive, zero, or positive imaginary in the respective cases above. The *light cone* is the set of light-like vectors, and its *interior* is the set of time-like vectors. A time-like or light-like vector is *positive* if its first entry is. The *hyperboloid model* $\mathbb{H}^n$ of hyperbolic space is the set of positive vectors with Lorentzian norm i in $\mathbb{R}^{n+1}$, equipped with the distance d_H defined by

$$\cosh d_H(\mathbf{u}, \mathbf{v}) = -\mathbf{u} \circ \mathbf{v}.$$

(We note that the following version of the Cauchy-Schwartz inequality follows from the usual one: for positive vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x}\circ\mathbf{x}\le 0$ and $\mathbf{y}\circ\mathbf{y}\le 0$, $\mathbf{x}\circ\mathbf{y}\le -\sqrt{(\mathbf{x}\circ\mathbf{x})(\mathbf{y}\circ\mathbf{y})}$, with equality if and only if they are linearly dependent, see eg. formula (1.0.2) of [1].)

The distance function d_H above is determined by the Riemannian metric on $\mathbb{H}^n$ given, at each $\mathbf{x}\in\mathbb{H}^n$, by restricting the Lorentzian inner product to $T_{\mathbf{x}}\mathbb{H}^n = \mathbf{x}^\perp \doteq \{\mathbf{v}\,|\,\mathbf{v}\circ\mathbf{x}=0\}$. (This restriction is positive-definite since $\mathbf{x}$ is time-like, see [4, Theorem 3.1.5].) The isometry group of $\mathbb{H}^n$ is the group $O^+(1,n)$ of matrices preserving the Lorentzian inner product and the sign of time-like vectors, see [4, §3.1], acting on $\mathbb{H}^n$ by restriction.

Given $\mathbf{x}\in\mathbb{H}^n$ and a *unit* space-like vector $\mathbf{y}$, ie. with $\mathbf{y}\circ\mathbf{y}=1$, if $\mathbf{y}\in T_{\mathbf{x}}\mathbb{H}^n$ (recall that this means $\mathbf{x}\circ\mathbf{y}=0$) then defining $\gamma_{\mathbf{y}}(t)=\cosh t\,\mathbf{x}+\sinh t\,\mathbf{y}$ determines a (unit-speed) geodesic in $\mathbb{H}^n$ with $\gamma_{\mathbf{y}}(0)=\mathbf{x}$ and $\gamma'_{\mathbf{y}}(0)=\mathbf{y}$. For an arbitrary $\mathbf{y}\in T_{\mathbf{x}}\mathbb{H}^n$,

$$\gamma_{\mathbf{y}}(t)\doteq\cosh\left(\|\mathbf{y}\|t\right)\,\mathbf{x}+\frac{1}{\|\mathbf{y}\|}\sinh\left(\|\mathbf{y}\|t\right)\,\mathbf{y} \tag{1}$$

is a constant-speed geodesic with $\gamma_{\mathbf{y}}(0)=\mathbf{x}$ and $\gamma'_{\mathbf{y}}(0)=\mathbf{y}$. (This can be directly checked.) The exponential map of $\mathbb{H}^n$ based at $\mathbf{x}$, a diffeomorphism $T_{\mathbf{x}}\mathbb{H}^n\to\mathbb{H}^n$, is then given by $\mathbf{y}\mapsto\gamma_{\mathbf{y}}(1)$.

The most useful feature of the hyperboloid model for us is that vectors of $\mathbb{R}^{n+1}$ which are not time-like also encode geometric features of $\mathbb{H}^n$.

1.1. **The meaning of light-like vectors.** Recall that $\mathbf{x}\in\mathbb{R}^{n+1}$ is *light-like* if $\mathbf{x}\circ\mathbf{x}=0$. Any positive light-like vector $\mathbf{x}$ is approached by a sequence of positive time-like vectors (for instance we can take $t\,\mathbf{x}+(1-t)\,\mathbf{e}_1$ for t approaching 1 from below); hence its projective class $[\mathbf{x}]$ in $\mathbb{R}P^n$ is approached by a sequence in the projectivization of $\mathbb{H}^n$. Conversely, the projectivization of the light cone is the frontier of the projectivization of $\mathbb{H}^n$ in $\mathbb{R}P^n$. In this sense we regard projectivized members of the light cone as *ideal points* of $\mathbb{H}^n$.

Individual vectors in the positive light cone carry more specific information.

Definition 1.1. The *horosphere determined by* a positive light-like vector $\mathbf{x}\in\mathbb{R}^{n+1}$ is $S=\{\mathbf{v}\in\mathbb{H}^n\,|\,\mathbf{v}\circ\mathbf{x}=-1\}$. The *horoball bounded by* S is the set $B=\{\mathbf{v}\in\mathbb{H}^n\,|\,\mathbf{v}\circ\mathbf{x}\ge -1\}$.

A little multivariable calculus shows that the horosphere S determined by a positive light-like vector $\mathbf{x}\in\mathbb{R}^{n+1}$ is the smooth submanifold $f^{-1}(-1)$ of $\mathbb{H}^n$, where $f(\mathbf{u})=\mathbf{u}\circ\mathbf{x}$, and its tangent space at any $\mathbf{u}_0\in S$ is $T_{\mathbf{u}_0}S=\{\mathbf{v}\in\mathbb{R}^{n+1}\,|\,\mathbf{v}\circ\mathbf{u}_0=0=\mathbf{v}\circ\mathbf{x}\}$. For any such $\mathbf{u}_0$ one may check directly that the formula $F(\mathbf{v})=\mathbf{u}_0+\mathbf{v}+\left(\frac{\mathbf{v}\circ\mathbf{v}}{2}\right)\mathbf{x}$ defines a Riemannian isometry from $T_{\mathbf{u}_0}S$, equipped with the restriction of the Lorentzian inner product, to $S\subset\mathbb{H}^n$. Since the inner product's restriction is positive-definite on $T_{\mathbf{u}_0}S$, this explicitly confirms the well known fact that S is an isometrically embedded copy of the Euclidean space $\mathbb{R}^{n-1}$. It also yields the following formula for the Euclidean distance $d_S(\mathbf{u}_0,\mathbf{u}_1)$ in S between vectors $\mathbf{u}_0$ and $\mathbf{u}_1$:

$$d_S(\mathbf{u}_0,\mathbf{u}_1)=\sqrt{-2(1+\mathbf{u}_0\circ\mathbf{u}_1)}$$

To see this, set $F(\mathbf{v})$ equal to $\mathbf{u}_1$ and solve for $\mathbf{v}\circ\mathbf{v}$ by taking the Lorentzian inner product of both sides with $\mathbf{u}_0$. Using the formula for $d_H(\mathbf{u}_0,\mathbf{u}_1)$ given above we obtain the comparison equation $d_S(\mathbf{u}_0,\mathbf{u}_1)/2=\sinh(d_H(\mathbf{u}_0,\mathbf{u}_1)/2)$. We note that this implies in particular that the isometric embedding F is proper; that is, S has compact intersection with any compact set of $\mathbb{H}^n$.

Lemma 1.2. *For $\mathbf{v}\in\mathbb{H}^n$ and a positive light-like vector $\mathbf{x}$, the signed hyperbolic distance d from $\mathbf{v}$ to the horosphere S determined by $\mathbf{x}$ satisfies $e^d=-\mathbf{v}\circ\mathbf{x}$, where the sign of d is positive if $\mathbf{v}$ lies outside the horoball B bounded by S. This distance is realized at $t=d$ on the geodesic*

$$\gamma(t)=e^{-t}\mathbf{v}-\frac{\sinh t}{\mathbf{x}\circ\mathbf{v}}\mathbf{x}=e^{-t}\mathbf{v}+e^{-d}\sinh t\,\mathbf{x}\quad\in\mathbb{H}^n,$$

which has $\gamma(0) = \mathbf{v}$. *We call* γ *the* geodesic through $\mathbf{v}$ in the direction of $\mathbf{x}$.

Proof. A vector $\mathbf{u} \in \mathbb{R}^{n+1}$ lies in S if and only if $\mathbf{u} \circ \mathbf{u} = -1$, so it lies in $\mathbb{H}^n$, and $\mathbf{u} \circ \mathbf{x} = -1$. By the theory of Lagrange multipliers, the restriction of $f(\mathbf{u}) \doteq \mathbf{u} \circ \mathbf{v}$ to S may attain a local extremum at $\mathbf{u} \in S$ only if the gradient of f at $\mathbf{u}$ is a linear combination of the gradients of the constraint functions $g_1(\mathbf{u}) \doteq \mathbf{u} \circ \mathbf{x}$ and $g_2(\mathbf{u}) \doteq \mathbf{u} \circ \mathbf{u}$. By a direct computation, $\nabla f(\mathbf{u}) = \bar{\mathbf{v}}$, $\nabla g_1(\mathbf{u}) = \bar{\mathbf{x}}$, and $\nabla g_2(\mathbf{u}) = \bar{\mathbf{u}}$, where $\bar{\mathbf{v}}$ is obtained from $\mathbf{v}$ by switching the sign of first entry, and similarly for the others. It follows that at any local extremum of the restriction of f to S, $\mathbf{v}$ is a linear combination of $\mathbf{x}$ and $\mathbf{u}$.

Since $\mathbf{v}$, which is time-like, is not a multiple of $\mathbf{x}$, which is light-like, this implies that we can express $\mathbf{u}$ in terms of $\mathbf{v}$ and $\mathbf{x}$. Upon plugging $\mathbf{u} = a\mathbf{x} + b\mathbf{v}$ into the constraints and solving for $a, b \in \mathbb{R}$ we obtain the unique solution

$$\mathbf{u} = \frac{1}{2}\left(1 - \frac{1}{(\mathbf{v} \circ \mathbf{x})^2}\right)\mathbf{x} - \frac{1}{\mathbf{v} \circ \mathbf{x}}\mathbf{v}. \tag{2}$$

The value of f at $\mathbf{u}$ is thus $\mathbf{u} \circ \mathbf{v} = \frac{1}{2}\left(\mathbf{v} \circ \mathbf{x} + \frac{1}{\mathbf{v} \circ \mathbf{x}}\right)$, so by the definition of the hyperbolic distance d_H we have

$$\cosh d_H(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\left(-\mathbf{v} \circ \mathbf{x} + \frac{1}{-\mathbf{v} \circ \mathbf{x}}\right).$$

Therefore $e^{d_H(\mathbf{u}, \mathbf{v})}$ is either $-\mathbf{v} \circ \mathbf{x}$ or its reciprocal, whichever is at least 1 since $d_H(\mathbf{u}, \mathbf{v})$ is non-negative. If we take d to be the *signed* distance, with non-negative sign if $\mathbf{v}$ is outside the horoball B, then by the definition of B we have $e^d = -\mathbf{v} \circ \mathbf{x}$ in all cases.

We finally note that d really is the (signed) distance from $\mathbf{v}$ to S; that is, the unique critical point $\mathbf{u}$ of f described above is the global maximizer for the values of f on S, so $d_H(\mathbf{x}, \mathbf{u})$ is the global minimizer of distances from $\mathbf{v}$ to points of S. This follows from uniqueness and the fact that as $\mathbf{u} \in \mathbb{H}^n$ escapes compact sets, $f(\mathbf{u}) \to -\infty$. Toward the latter point, note for an arbitrary $\mathbf{u} = (u_1, \dots, u_{n+1}) \in \mathbb{H}^n$ that $u_1 = \sqrt{1 + u_2^2 + \ldots + u_{n+1}^2}$, so we can rewrite $f(\mathbf{u})$ as

$$\begin{aligned} f(\mathbf{u}) &= -\sqrt{(1 + u_2^2 + \ldots + u_{n+1}^2)(1 + v_2^2 + \ldots + v_{n+1}^2)} + u_2 v_2 + \ldots + u_{n+1} v_{n+1} \\ &= \frac{(u_2 v_2 + \ldots + u_{n+1} v_{n+1})^2 - (1 + u_2^2 + \ldots + u_{n+1}^2)(1 + v_2^2 + \ldots + v_{n+1}^2)}{\sqrt{(1 + u_2^2 + \ldots + u_{n+1}^2)(1 + v_2^2 + \ldots + v_{n+1}^2)} + u_2 v_2 + \ldots + u_{n+1} v_{n+1}}. \end{aligned}$$

In passing from the first to the second line above we use the fact that $\sqrt{a} - \sqrt{b} = (a - b)/(\sqrt{a} + \sqrt{b})$. Expanding the numerator, canceling certain terms, and rearranging yields:

$$-1 - (u_2^2 + \ldots + u_{n+1}^2) - (v_2^2 + \ldots + v_{n+1}^2) - \sum_{i \neq j} (u_i - v_j)^2.$$

The denominator is at most some fixed multiple of $\sqrt{1 + u_2^2 + \ldots + u_{n+1}^2}$, by the Cauchy-Schwarz inequality, whereas the numerator is at most the opposite of the square of this quantity. So as claimed, $f(\mathbf{u}) \to -\infty$ as $\mathbf{u}$ escapes compact sets.

For the parametrized curve γ defined in the statement, direct computation reveals that $\gamma(t) \circ \gamma(t) = -1$ for all t, so γ maps into $\mathbb{H}^n$, and that $\gamma''(t) = \gamma(t)$. Therefore γ is a hyperbolic geodesic, by [4, Theorem 3.2.4]. More direct computation shows that $\gamma(0) = \mathbf{v}$ and $\gamma(d)$ is the nearest point $\mathbf{u}$ to $\mathbf{v}$ on S described in (2). $\square$

Lemma 1.3. *For linearly independent positive light-like vectors* $\mathbf{x}_0$ *and* $\mathbf{x}_1$ *of* $\mathbb{R}^{n+1}$*, the minimum signed distance* d *from points on* S_1 *to* S_0 *satisfies* $e^d = -\frac{1}{2}\mathbf{x}_0 \circ \mathbf{x}_1$*, where* S_i *is the horosphere* *of* $\mathbb{H}^n$ *determined by* $\mathbf{x}_i$ *for* $i = 0, 1$*. This distance is uniquely attained by points at* $t = \pm d/2$ *on the geodesic*

$$\gamma(t) = \frac{1}{\sqrt{-2(\mathbf{x}_0 \circ \mathbf{x}_1)}} \left(e^t\,\mathbf{x}_0 + e^{-t}\,\mathbf{x}_1\right) = \frac{1}{2}e^{-d/2}\left(e^t\,\mathbf{x}_0 + e^{-t}\,\mathbf{x}_1\right)$$

from $\mathbf{x}_1$ *to* $\mathbf{x}_0$*.*

Proof. A vector $\mathbf{u} \in \mathbb{R}^{n+1}$ lies in S_1 if and only if $\mathbf{u} \circ \mathbf{u} = -1$, $\mathbf{u}$ is positive, and $\mathbf{u} \circ \mathbf{x}_1 = -1$. By the theory of Lagrange multipliers, the restriction of $f(\mathbf{u}) = \mathbf{u} \circ \mathbf{x}_0$ to B_1 may attain a local extremum at $\mathbf{u} \in S_1$ only if the gradient of f at $\mathbf{u}$ is a linear combination of the constraint gradients $\nabla g_1(\mathbf{u})$ and $\nabla g_2(\mathbf{u})$, where $g_1(\mathbf{u}) = \mathbf{u} \circ \mathbf{x}_1$ and $g_2(\mathbf{u}) = \mathbf{u} \circ \mathbf{u}$. Direct computation yields $\nabla f(\mathbf{u}) = \bar{\mathbf{x}}_0$, $\nabla g_1(\mathbf{u}) = \bar{\mathbf{x}}_1$, and $\nabla g_2(\mathbf{u}) = 2\bar{\mathbf{u}}$, where $\bar{\mathbf{x}}_0$ is obtained from $\mathbf{x}_0$ by multiplying the first entry by -1 and similarly for the others. It thus follows that at such a local extremum $\mathbf{u}$, $\mathbf{x}_0$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{u}$ so, since $\mathbf{x}_0$ is not a multiple of $\mathbf{x}_1$, $\mathbf{u}$ is a linear combination of the $\mathbf{x}_i$.

Plugging $\mathbf{u} = a\mathbf{x}_0 + b\mathbf{x}_1$ into the constraint equations and solving for $a, b \in \mathbb{R}$ yields

$$\mathbf{u} = \frac{-1}{\mathbf{x}_0 \circ \mathbf{x}_1}\mathbf{x}_0 + \frac{1}{2}\mathbf{x}_1 \tag{3}$$

This is a positive vector since it is a positive linear combination of the positive vectors $\mathbf{x}_0$ and $\mathbf{x}_1$. By Lemma 1.2 and a direct computation, the signed distance d from $\mathbf{u}$ to S_0 satisfies $e^d = -\frac{1}{2}\mathbf{x}_0 \circ \mathbf{x}_1$.

Substituting $\mathbf{u}$ for $\mathbf{v}$ in the formula for the geodesic $\gamma(t)$ defined in Lemma 1.2 and simplifying yields

$$\gamma(t) = \frac{e^t}{-\mathbf{x}_0 \circ \mathbf{x}_1}\mathbf{x}_0 + \frac{e^{-t}}{2}\mathbf{x}_1.$$

Note that $\gamma(0) = \mathbf{u} \in S_1$ and $\gamma(d) \in S_0$. The more-symmetric formula given in the statement is obtained by translating the parametrization, replacing t by $t - d/2$.

It remains to show for $\mathbf{u}$ from the formula (3) that $f(\mathbf{u})$ is a global maximum of f on S_1, hence that d is a global minimum of the signed distance to S_0 on S_1. This follows from the fact that $\mathbf{u}$ is the unique critical point of f on S_1, together with the fact that $f(\mathbf{v}) \to -\infty$ as $\mathbf{v} \in S_1$ escapes compact sets. Indeed, for any fixed $r < 0$, and any $\mathbf{v} \in S_1$ such that $f(\mathbf{v}) \geq r$, we have $\mathbf{v} \circ \mathbf{u} = -f(\mathbf{v})/\mathbf{x}_0 \circ \mathbf{x}_1 - 1/2 \geq -r/\mathbf{x}_0 \circ \mathbf{x}_1 - 1/2$, so $\mathbf{v}$ is contained in the closed ball of radius $\cosh^{-1}(r/\mathbf{x}_0 \circ \mathbf{x}_1 + 1/2)$ around $\mathbf{u}$. This ball is compact. □

1.2. **The meaning of space-like vectors.** Recall that $\mathbf{y} \in \mathbb{R}^{n+1}$ is *space-like* if $\mathbf{y} \circ \mathbf{y} > 0$. We note that the orthogonal subspace $V = \{\mathbf{x} \circ \mathbf{y} = 0\}$ to a space-like vector $\mathbf{y}$ is time-like, ie. containing a time-like vector, since if this were not so then $\mathbb{R}^{n+1}$ would have no time-like vectors. This motivates:

Definition 1.4. The *polar hyperplane* to a space-like vector $\mathbf{y}$ is $P = \{\mathbf{x} \in \mathbb{H}^n \mid \mathbf{x} \circ \mathbf{y} = 0\}$.

As defined in [4, §3.2], a *hyperplane* of $\mathbb{H}^n$ is its intersection with a time-like, codimension-one vector subspace of $\mathbb{R}^{n+1}$. Corollary 4 of [4, §3.2] implies that the group of hyperbolic isometries acts transitively on the set of hyperplanes. Thus each hyperplane is the polar hyperplane to a space-like vector, since for instance $(\mathbb{R}^n \times \{0\}) \cap \mathbb{H}^n$ is the polar hyperplane to $\mathbf{e}_{n+1} = (0, \ldots, 0, 1)$.

Every hyperplane $P = V \cap \mathbb{H}^n$ is a totally geodesic copy of $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$, being, for any $\mathbf{x} \in P$, the image of the restriction of the exponential map based at $\mathbf{x}$ to $T_{\mathbf{x}}P = V \cap \mathbf{x}^{\perp}$. Conversely, the exponential map's explicit description shows that any $(n-1)$-dimensional totally geodesic subspace P of $\mathbb{H}^n$ is contained in $V = \operatorname{span}\{\mathbf{x}, T_{\mathbf{x}}P\}$ for any $\mathbf{x} \in P$, and hence is a hyperplane.

We define a *half-space* to be the closure of one component of $\mathbb{H}^n - P$, for a hyperplane P. We call P the *boundary* of H and $H - P$ the *interior*. From eg. the model case above we see that each hyperplane bounds exactly two distinct half-spaces, which have disjoint interiors.

Lemma 1.5. *There is a bijective correspondence between half-spaces of $\mathbb{H}^n$ and unit space-like vectors of $\mathbb{R}^{n+1}$ that sends $\mathbf{y} \in \mathbb{R}^{n+1}$ to $H = \{\mathbf{x} \in \mathbb{H}^n \,|\, \mathbf{x} \circ \mathbf{y} \leq 0\}$. In the other direction, it sends a half-space H to the unit outward normal $\mathbf{y}$ to H at any point of its boundary.*

Above, given a hyperplane P and any $\mathbf{x} \in P$, a *normal* vector to P at $\mathbf{x}$—and to a half-space H bounded by P—is an element of $T_{\mathbf{x}}\mathbb{H}^n$ orthogonal to the codimension-one subspace $T_{\mathbf{x}}P$. A unit normal vector $\mathbf{y}$ to P determines a geodesic $\gamma_{\mathbf{y}}(t) = \cosh t\, \mathbf{x} + \sinh t\, \mathbf{y}$ that intersects P transversely, and we say $\mathbf{y}$ is *outward* to H if $\gamma(t) \in H$ for all $t < 0$.

Proof. For a hyperplane P and any $\mathbf{x} \in P$, since the orthogonal subspace to $T_{\mathbf{x}}P$ in $T_{\mathbf{x}}\mathbb{H}^n$ is one-dimensional there are exactly two unit normals to P. If $\mathbf{y}$ is one of these, the other is $-\mathbf{y}$, and exactly one of them is outward to a given half-space H bounded by P. Take this to be $\mathbf{y}$. Any $\mathbf{x}' \in P$ is of the form $\gamma_{\mathbf{z}}(1)$ for some $\mathbf{z} \in T_{\mathbf{x}}P$, with $\gamma_{\mathbf{z}}$ as in (1)—ie. $\mathbf{x}'$ is the exponential image of $\mathbf{z}$—and hence $\mathbf{y} \circ \mathbf{x}'$ also equals 0. Thus P is the polar hyperplane of $\mathbf{y}$.

For this $\mathbf{y}$, we claim that $H = \{\mathbf{x} \in \mathbb{H}^n \,|\, \mathbf{x} \circ \mathbf{y} < 0\}$. Defining $f\colon \mathbb{H}^n \to \mathbb{R}$ by $f(\mathbf{x}) = \mathbf{x} \circ \mathbf{y}$, note that since the interior of H is a connected component of the complement of $P = f^{-1}(0)$, it maps into one of $(-\infty, 0)$ or $(0, \infty)$ under f. Since it contains $\gamma_{\mathbf{y}}(t)$ for $t < 0$, it is the former. Similarly, the other component of $\mathbb{H}^n - P$ maps into $(0, \infty)$, so the claim holds.

Conversely, a unit space-like vector $\mathbf{y}$ belongs to $T_{\mathbf{x}}\mathbb{H}^n = \mathbf{x}^\perp$ at any point $\mathbf{x}$ of its polar hyperplane P, and it is normal to $T_{\mathbf{x}}P = V \cap \mathbf{x}^\perp$ for $V = \{\mathbf{v} \in \mathbb{R}^{n+1} \,|\, \mathbf{v} \circ \mathbf{x} = 0\}$. A computation shows that it is also the outward normal to the half-space $H = \{\mathbf{x} \in \mathbb{H}^n \,|\, \mathbf{x} \circ \mathbf{y} \leq 0\}$. □

We use this to give a series of geometric interpretations on the Lorentz pairing between vectors of various types and space-like vectors. The first follows directly from Theorem 3.2.12 of [4].

Lemma 1.6. *For $\mathbf{v} \in \mathbb{H}^n$ and a unit space-like vector $\mathbf{y}$, the signed distance d from $\mathbf{v}$ to the polar hyperplane of $\mathbf{y}$ satisfies $\sinh d = \mathbf{v} \circ \mathbf{y}$, where the sign is negative if and only if $\mathbf{v}$ is contained in the interior of the half-space bounded by P with outward normal $\mathbf{y}$.*

In the next result and below, the *ideal boundary* of a hyperplane $P = V \cap \mathbb{H}^n$ (respectively, a half-space H bounded by P) is the intersection of V (resp. the closure of the component of $\mathbb{R}^{n+1} - V$ containing the interior of H) with the positive light cone.

Lemma 1.7. *For a positive light-like vector $\mathbf{x} \in \mathbb{R}^{n+1}$, let S be the horosphere determined by $\mathbf{x}$. Suppose $P \subset \mathbb{H}^n$ is a hyperplane with ideal boundary not containing $\mathbf{x}$, and let $\mathbf{y} \in \mathbb{R}^{n+1}$ be the outward-pointing normal to the half-space H bounded by P with ideal boundary containing $\mathbf{x}$. Then $\mathbf{x} \circ \mathbf{y} < 0$, and the minimal signed distance h from P to S satisfies $e^h = -\mathbf{x} \circ \mathbf{y}$, uniquely realized by $\gamma(0) \in P$ and $\gamma(h) \in S$ for*

$$\gamma(t) = e^{-h} \cosh t\, \mathbf{x} + e^{-t}\, \mathbf{y}.$$

This is the unique geodesic perpendicular to P in the direction of $\mathbf{x}$, in the sense of Lemma 1.2.

Proof. A vector $\mathbf{v} \in \mathbb{R}^{n+1}$ lies in P if and only if $\mathbf{v} \circ \mathbf{v} = -1$, $\mathbf{v}$ is positive, and $\mathbf{v} \circ \mathbf{y} = 0$. By the theory of Lagrange multipliers, the restriction of $f(\mathbf{v}) \doteq \mathbf{v} \circ \mathbf{x}$ to P may attain a local extremum at $\mathbf{v} \in P$ only if the gradient of f at $\mathbf{v}$ is a linear combination of the constraint gradients $\nabla g_1(\mathbf{v})$ and $\nabla g_2(\mathbf{v})$, where $g_1(\mathbf{v}) = \mathbf{v} \circ \mathbf{y}$ and $g_2(\mathbf{v}) = \mathbf{v} \circ \mathbf{v}$. Direct computation yields $\nabla f(\mathbf{v}) = \bar{\mathbf{x}}$, $\nabla g_1(\mathbf{v}) = \bar{\mathbf{y}}$, and $\nabla g_2(\mathbf{v}) = 2\bar{\mathbf{v}}$, where $\bar{\mathbf{x}}$ is obtained from $\mathbf{x}$ by multiplying the first entry by -1 and similarly for the others. It thus follows that $\mathbf{x}$ is a linear combination of $\mathbf{y}$ and $\mathbf{v}$ for such a point $\mathbf{v}$, so since $\mathbf{x}$ is not a multiple of $\mathbf{y}$ we can express $\mathbf{v}$ in terms of $\mathbf{x}$ and $\mathbf{y}$.

Plugging $\mathbf{v} = a\mathbf{x} + b\mathbf{y}$ into the constraint equations and solving for $a, b \in \mathbb{R}$ yields:

$$\mathbf{v} = \pm \left(\frac{-1}{\mathbf{x} \circ \mathbf{y}} \mathbf{x} + \mathbf{y} \right) \tag{4}$$

Only one of these two solutions is positive. We claim that $\mathbf{v}$ is positive and hence is the unique critical point of the restriction of f to H. By Lemma 1.2 its signed distance h to B will then satisfy $e^h = -\mathbf{x} \circ \mathbf{y}$, and the geodesic through $\mathbf{v}$ in the direction of $\mathbf{x}$ will be given by:

$$\gamma(t) = e^{-t}\mathbf{v} - \frac{\sinh t}{\mathbf{x} \circ \mathbf{v}}\mathbf{x} = \frac{\cosh t}{-\mathbf{x} \circ \mathbf{y}}\mathbf{x} + e^{-t}\mathbf{y} = e^{-h}\cosh t\,\mathbf{x} + e^{-t}\,\mathbf{y}.$$

To prove the claim, we first note that $\mathbf{x} \circ \mathbf{y} < 0$: this follows from the fact that the half-space H whose ideal boundary contains $\mathbf{x}$ is characterized as $H = \{\mathbf{v} \in \mathbb{H}^n \,|\, \mathbf{v} \circ \mathbf{y} \leq 0\}$. We then write $\mathbf{x} = (x_1, \mathbf{x}_0)$ and $\mathbf{y} = (y_1, \mathbf{y}_0)$ for vectors $\mathbf{x}_0, \mathbf{y}_0 \in \mathbb{R}^n$, so the first entry of $\mathbf{v}$ is $x_1/(-\mathbf{x} \circ \mathbf{y}) + y_1$. The hypothesis that $\mathbf{x}$ is positive means that $x_1 > 0$, so since $\mathbf{x} \circ \mathbf{y} < 0$, the first entry of $\mathbf{v}$ is certainly positive if $y_1 \geq 0$. We therefore suppose that $y_1 < 0$. Since $\mathbf{x}$ is light-like and $\mathbf{y}$ is unit space-like, we can write $x_1 = \|\mathbf{x}_0\|$ and $y_1 = -\sqrt{\|\mathbf{y}_0\|^2 - 1}$, and hence

$$\mathbf{x} \circ \mathbf{y} = \|\mathbf{x}_0\|\sqrt{\|\mathbf{y}_0\|^2 - 1} + \mathbf{x}_0 \cdot \mathbf{y}_0,$$

where $\mathbf{x}_0 \cdot \mathbf{y}_0$ is the ordinary dot product of $\mathbf{x}_0$ and $\mathbf{y}_0$. Since $\mathbf{x} \circ \mathbf{y} < 0$ we must have $\mathbf{x}_0 \cdot \mathbf{y}_0 < 0$; by the Cauchy Schwarz inequality, $-\mathbf{x}_0 \cdot \mathbf{y}_0 \leq \|\mathbf{x}_0\|\|\mathbf{y}_0\|$. Thus we have:

$$\begin{aligned} \frac{-1}{\mathbf{x} \circ \mathbf{y}} x_1 + y_1 &= \frac{\|\mathbf{x}_0\|}{-\mathbf{x}_0 \cdot \mathbf{y}_0 - \|\mathbf{x}_0\|\sqrt{\|\mathbf{y}_0\|^2 - 1}} - \sqrt{\|\mathbf{y}_0\|^2 - 1} \\ &\geq \frac{\|\mathbf{x}_0\|}{\|\mathbf{x}_0\|\|\mathbf{y}_0\| - \|\mathbf{x}_0\|\sqrt{\|\mathbf{y}_0\|^2 - 1}} - \sqrt{\|\mathbf{y}_0\|^2 - 1} \end{aligned}$$

Simplifying the above and using the fact that $1/(\|\mathbf{y}_0\| - \sqrt{\|\mathbf{y}_0\|^2 - 1}) = \|\mathbf{y}_0\| + \sqrt{\|\mathbf{y}_0\|^2 - 1}$, we obtain in this case that $x_1/(-\mathbf{x} \circ \mathbf{y}) + y_1 \geq \|\mathbf{y}_0\| > 0$. This proves the claim.

It remains to show that $\mathbf{v}$ is the global maximizer for the restriction of f to P, hence that it is the minimizer for the signed distance to S. This follows from the fact that $\mathbf{v}$ is the unique critical point of the restriction of f to P, together with the fact that $f(\mathbf{u}) \to -\infty$ as $\mathbf{u} \in P$ escapes compact sets. Indeed, for any fixed $r < 0$ and $\mathbf{u} \in P$ such that $\mathbf{u} \circ \mathbf{x} > r$, we have $\mathbf{u} \circ \mathbf{v} = (-1/\mathbf{x} \circ \mathbf{y})\mathbf{u} \circ \mathbf{x} > -r/\mathbf{x} \circ \mathbf{y}$; hence $\mathbf{u}$ lies in the closed ball of radius $\cosh^{-1}(r/\mathbf{x} \circ \mathbf{y})$ about $\mathbf{v}$. □

The result below combines a few recorded by Ratcliffe in [4].

Lemma 1.8 (cf. [4], pp. 65–69)**.** *Let $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^{n+1}$ be linearly independent space-like vectors, with polar hyperplanes P_1 and P_2 in $\mathbb{H}^n$, contained in n-dimensional subspaces V_1 and V_2 of $\mathbb{R}^{n+1}$, respectively. Exactly one of the following holds:*

(1) P_1 and P_2 intersect in $\mathbb{H}^n$, and $|\mathbf{y}_1 \circ \mathbf{y}_2| < \|\mathbf{y}_1\|\|\mathbf{y}_2\|$. Hence for some $\eta(\mathbf{y}_1, \mathbf{y}_2) \in (0, \pi)$:

$$\mathbf{y}_1 \circ \mathbf{y}_2 = \|\mathbf{y}_1\|\|\mathbf{y}_2\| \cos\eta(\mathbf{y}_1, \mathbf{y}_2).$$

For any $\mathbf{v} \in P_1 \cap P_2$, $\eta(\mathbf{y}_1, \mathbf{y}_2)$ is the angle in $T_{\mathbf{v}}\mathbb{H}^n$ between the normal vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ to P_1 and P_2, respectively, at $\mathbf{v}$.

(2) The distance between points of P_1 and P_2 attains a non-zero minimum, and $|\mathbf{y}_1 \circ \mathbf{y}_2| > \|\mathbf{y}_1\|\|\mathbf{y}_2\|$. Hence for some $\eta(\mathbf{y}_1, \mathbf{y}_2) \in (0, \infty)$:

$$|\mathbf{y}_1 \circ \mathbf{y}_2| = \|\mathbf{y}_1\|\|\mathbf{y}_2\| \cosh\eta(\mathbf{y}_1, \mathbf{y}_2).$$

In this case $\eta(\mathbf{y}_1, \mathbf{y}_2)$ is the (minimum) distance in $\mathbb{H}^n$ between P_1 and P_2, and $\mathbf{y}_1 \circ \mathbf{y}_2 < 0$ if and only if $\mathbf{y}_1$ and $\mathbf{y}_2$ are oppositely oriented tangent vectors to the hyperbolic geodesic intersecting each of P_1 and P_2 perpendicularly.

(3) $P_1 \cap P_2 = \emptyset$ *but their ideal boundaries intersect, and* $|\mathbf{y}_1 \circ \mathbf{y}_2| = \|\mathbf{y}_1\|\|\mathbf{y}_2\|$.

In case (3) above we say that P_1 and P_2 are *parallel*. One can show in this case that there are sequences in P_1 and P_2 such that the infimum of distances from points of the first sequence to points of the second is 0. We now expand on case (2) above.

Lemma 1.9. *Suppose* $\mathbf{y}_1$ *and* $\mathbf{y}_2$ *are linearly independent space-like vectors such that the distance between points of their polar hyperplanes* P_1 *and* P_2 *attains a non-zero minimum. This distance is realized as* $d(\mathbf{v}_1, \mathbf{v}_2)$ *for unique* $\mathbf{v}_1 \in P_1$ *and* $\mathbf{v}_2 \in P_2$*, with* $\mathbf{v}_1$ *given by:*

$$\mathbf{v}_1 = \pm \frac{(\mathbf{y}_1 \circ \mathbf{y}_2 / \|\mathbf{y}_1\|)\, \mathbf{y}_1 - \|\mathbf{y}_1\| \mathbf{y}_2}{\sqrt{(\mathbf{y}_1 \circ \mathbf{y}_2)^2 - \|\mathbf{y}_1\|^2 \|\mathbf{y}_2\|^2}},$$

where the sign of "$\pm$" above is negative if $\mathbf{v}_1$ *belongs to the half-space* H_2 *bounded by* P_2 *with* $\mathbf{y}_2$ *as outward unit normal vector, and negative otherwise.*

Proof. Standard facts of hyperbolic geometry imply the uniqueness of $\mathbf{v}_1 \in P_1$ and $\mathbf{v}_2 \in P_2$, and furthermore that the geodesic γ joining $\mathbf{v}_1$ and $\mathbf{v}_2$ intersects each of P_1 and P_2 perpendicularly. Therefore γ has tangent vector $\mathbf{y}_1$ at $\mathbf{v}_1$ and $\mathbf{y}_2$ at $\mathbf{v}_2$, and it follows that $\gamma = \mathrm{Span}\{\mathbf{y}_1, \mathbf{y}_2\} \cap \mathbb{H}^n$. Taking $\mathbf{v}_1 = a\mathbf{y}_1 + b\mathbf{y}_2$ and solving the equations $\mathbf{v}_1 \circ \mathbf{y}_1 = 0$ and $\mathbf{v}_1 \circ \mathbf{v}_1 = -1$ (necessary for $\mathbf{v}_1 \in \mathbb{H}^n$) for a and b yields the two solutions above. Taking an inner product with $\mathbf{y}_2$ now yields

$$\mathbf{v}_1 \circ \mathbf{y}_2 = \pm \frac{1}{\|\mathbf{y}_1\|} \frac{(\mathbf{y}_1 \circ \mathbf{y}_2)^2 - \|\mathbf{y}_1\|^2 \|\mathbf{y}_2\|^2}{\sqrt{(\mathbf{y}_1 \circ \mathbf{y}_2)^2 - \|\mathbf{y}_1\|^2 \|\mathbf{y}_2\|^2}}$$

By Lemma 1.5, $\mathbf{v}_1$ belongs to the half-space H_2 with $\mathbf{y}_2$ as outward normal if and only if $\mathbf{v}_1 \circ \mathbf{y}_2 < 0$, hence if and only if the "$\pm$" above is negative. □

2. Dimension two

Here we prove trigonometric formulas for a hyperbolic quadrilateral with two ideal vertices and a hyperbolic pentagon with one ideal vertex, each with right angles at all finite vertices.

Proposition 2.1. *Let* $Q \subset \mathbb{H}^2$ *be a convex quadrilateral with a single compact side of length* ℓ *and right angles at its endpoints, and let* B_0 *and* B_1 *be horoballs centered at the two ideal vertices of* Q*. If* a_i *is the signed distance to* B_i *from the other endpoint of the half-open edge of* Q *containing the ideal point of* B_i*,* $i = 0, 1$*, and* d *is the signed distance from* B_0 *to* B_1*, then*

$$\sinh(\ell/2) = e^{(d - a_0 - a_1)/2}.$$

If θ_i *is the length of the horocyclic arc* $S_i \cap Q$*,* $i = 0, 1$*, where* $S_i = \partial B_i$*, then for each* i*,*

$$\frac{\theta_0}{e^{a_1}} = \frac{\theta_1}{e^{a_0}} = \frac{\sinh \ell}{2e^d}$$

Proof. For a quadrilateral $Q \subset \mathbb{H}^2$ with a single compact edge γ and right angles at the endpoints of this edge, let $\mathbf{x}_0$ and $\mathbf{x}_1$ be positive light-like vectors determining the horobolls B_0 and B_1 centered at the ideal vertices of Q. Using the fact that the geodesic containing γ is a codimension-one hyperplane of $\mathbb{H}^2$, let $\mathbf{y}$ be the space-like vector Lorentz-orthogonal to this geodesic with the property that $\mathbf{x}_i \circ \mathbf{y} < 0$ for $i = 0, 1$. (Since the ideal vertices of Q are on the same side of this geodesic, the inner products with $\mathbf{y}$ have the same sign by Lemma 1.7.)

Let $\mathbf{v}_0$ and $\mathbf{v}_1$ be the finite vertices of Q, numbered so that $\mathbf{v}_i$ is an endpoint of the half-open edge of Q with its other endpoint at the center of B_i, for $i = 0, 1$. Since Q is right-angled, $\mathbf{v}_i$ is described in terms of $\mathbf{x}_i$ and $\mathbf{y}$ by the formula (4) for each i. (Note that there is a unique geodesic ray perpendicular to the geodesic containing γ with its ideal endpoint at the center of B_i, since there is no hyperbolic triangle with two right angles.) That is:

$$\mathbf{v}_0 = \frac{-1}{\mathbf{x}_0 \circ \mathbf{y}}\mathbf{x}_0 + \mathbf{y} \qquad\qquad \mathbf{v}_1 = \frac{-1}{\mathbf{x}_1 \circ \mathbf{y}}\mathbf{x}_1 + \mathbf{y}$$

By Lemma 1.7 their signed distances a_i to the B_i satisfy $e^{a_i} = -\mathbf{x}_i \circ \mathbf{y}$ for $i = 0, 1$. If ℓ is the length of γ then from the distance formula we obtain

$$\cosh \ell = -\mathbf{v}_1 \circ \mathbf{v}_2 = \frac{-\mathbf{x}_0 \circ \mathbf{x}_1}{(\mathbf{x}_0 \circ \mathbf{y})(\mathbf{x}_1 \circ \mathbf{y})} + 1$$

It follows from Lemma 1.3 that the minimal signed distance d from S_0 to S_1 satisfies $e^d = -\frac{1}{2}\mathbf{x}_0 \circ \mathbf{x}_1$, hence by a half-angle formula $\sinh(\ell/2) = e^{(d-a_0-a_1)/2}$ as claimed.

Let $\mathbf{u}_0$ and $\mathbf{u}_0'$ be the points of intersection between the horosphere $S_0 = \partial B_0$ and the edges of Q joining the class of $\mathbf{x}_0$ to $\mathbf{v}_0$ and the class of $\mathbf{x}_1$, respectively. We obtain an explicit description for $\mathbf{u}_0$ by plugging in $t = a_0$ to the parametrized geodesic $\gamma(t)$ starting at $\mathbf{v}_0$ given in Lemma 1.7, and for $\mathbf{u}_0'$ by plugging in $t = d/2$ to the parametrized geodesic $\lambda(t)$ from $\mathbf{x}_1$ given in Lemma 1.3. These yield:

$$\mathbf{u}_0 = \frac{1}{2}\left(1 + \frac{1}{(\mathbf{x}_0 \circ \mathbf{y})^2}\right)\mathbf{x}_0 + \frac{-1}{\mathbf{x}_0 \circ \mathbf{y}}\mathbf{y} \qquad\qquad \mathbf{u}_0' = \frac{1}{2}\mathbf{x}_0 + \frac{-1}{\mathbf{x}_0 \circ \mathbf{x}_1}\mathbf{x}_1$$

From the horospherical distance formula we thus have

$$\theta_0 = d_{S_0}(\mathbf{u}_0, \mathbf{u}_0') = \sqrt{-2(1 + \mathbf{u}_0 \circ \mathbf{u}_0')} = \sqrt{\frac{1}{(\mathbf{x}_0 \circ \mathbf{y})^2} - \frac{2(\mathbf{x}_1 \circ \mathbf{y})}{(\mathbf{x}_0 \circ \mathbf{x}_1)(\mathbf{x}_0 \circ \mathbf{y})}}$$

A similar computation yields an analogous formula for θ_1, and we observe that

$$\begin{aligned}\theta_0 e^{-a_1} = \theta_1 e^{-a_0} &= \sinh \ell/(2e^d) \\ &= \frac{1}{(\mathbf{x}_0 \circ \mathbf{y})(\mathbf{x}_1 \circ \mathbf{y})}\sqrt{\frac{2(\mathbf{x}_0 \circ \mathbf{y})(\mathbf{x}_1 \circ \mathbf{y}) - \mathbf{x}_0 \circ \mathbf{x}_1}{-\mathbf{x}_0 \circ \mathbf{x}_1}}\end{aligned}$$

The latter assertion in the statement follows. □

Proposition 2.2. *Let $P \subset \mathbb{H}^2$ be a convex pentagon with four right angles and one ideal vertex, and let B be a horoball centered at the ideal vertex of P. Let d be the length of the side of P opposite its ideal vertex, let $\mathbf{w}_0$ and $\mathbf{w}_1$ be its endpoints, and for $i = 0, 1$ let ℓ_i be the length of the other side containing $\mathbf{w}_i$. If $\mathbf{v}_i$ is the other endpoint of this side and a_i is its signed distance to B, for $i = 0, 1$, then*

$$\cosh \ell_i = \frac{e^{a_i}\cosh d + e^{a_{1-i}}}{e^{a_i}\sinh d} \quad \textit{for } i = 0, 1.$$

Moreover, if θ is the length of the horocyclic arc $S \cap P$, where $S = \partial B$, then

$$\frac{\theta}{\sinh d} = \frac{\sinh \ell_0}{e^{a_1}} = \frac{\sinh \ell_1}{e^{a_0}}.$$

Proof. Let P be a pentagon with four right angles and a single ideal vertex, and let $\mathbf{x}$ be a positive light-like vector that determines a horosphere S centered at the ideal vertex of P. Labeling the endpoints of the edge of P opposite its ideal vertex as $\mathbf{w}_0$ and $\mathbf{w}_1$, for $i = 0, 1$ let γ_i be the other edge of P containing $\mathbf{w}_i$, and let $\mathbf{y}_i$ be a unit space-like vector in $\mathbb{R}^3$ orthogonal to the geodesic containing γ_i. Choose the $\mathbf{y}_i$ so that $\mathbf{y}_i \circ \mathbf{x} < 0$ for each i. Equivalently, by Lemma 1.7, $\mathbf{y}_i$ is on the opposite side of $\mathbf{x}$ from the plane $\mathbf{u} \circ \mathbf{x} = 0$ in $\mathbb{R}^3$. Since γ_i and the ideal point of P are on the same side of the geodesic containing γ_{1-i} for each i, $\mathbf{y}_0 \circ \mathbf{y}_1 < 0$ by [4].

Let us call $\mathbf{v}_i$ the endpoint of γ_i not equal to $\mathbf{w}_i$, for $i = 0, 1$. An explicit formula for $\mathbf{v}_i$ is given by (4), with $\mathbf{y}$ there replaced by $\mathbf{y}_i$. As in the proofs of Theorem 3.2.7 and 3.2.8 of [4] we have the following explicit formula for $\mathbf{w}_i$:

$$\mathbf{w}_i = \frac{-(\mathbf{y}_0 \circ \mathbf{y}_1)\mathbf{y}_i + \mathbf{y}_{i-1}}{\pm\sqrt{(\mathbf{y}_0 \circ \mathbf{y}_1)^2 - 1}},$$

where "+" or "−" is chosen so that $\mathbf{w}_i$ is a positive vector. For, say, $i = 0$ we thus have

$$\mathbf{w}_0 \circ \mathbf{v}_0 = \frac{\mathbf{y}_0 \circ \mathbf{y}_1 - (\mathbf{x} \circ \mathbf{y}_1)/(\mathbf{x} \circ \mathbf{y}_0)}{\pm\sqrt{(\mathbf{y}_0 \circ \mathbf{y}_1)^2 - 1}} = \frac{-(\mathbf{x} \circ \mathbf{y}_0)(\mathbf{y}_0 \circ \mathbf{y}_1) + \mathbf{x} \circ \mathbf{y}_1}{-(\mathbf{x} \circ \mathbf{y}_0)\sqrt{(\mathbf{y}_0 \circ \mathbf{y}_1)^2 - 1}}$$

In passing from the first to the second equality we have fixed the sign choice "+" for the radical. This is the right choice since $\mathbf{y}_0 \circ \mathbf{y}_1$ and the $\mathbf{x} \circ \mathbf{y}_i$ are all negative, and $\mathbf{w}_0 \circ \mathbf{v}_0$ is as well.

If ℓ_i is the length of γ_i and a_i is the distance from $\mathbf{v}_i$ to S, for $i = 0, 1$, and $d = d_H(\mathbf{w}_0, \mathbf{w}_1)$ is the length of the side opposite the ideal vertex, then the above equation becomes

$$\cosh \ell_0 = \frac{e^{a_0} \cosh d + e^{a_1}}{e^{a_0} \sinh \ell}$$

This is because $\cosh \ell = -\mathbf{w}_0 \circ \mathbf{v}_0$ by definition, $d_H(\mathbf{v}_i, S) = -\mathbf{x} \circ \mathbf{y}_i$ by Lemma 1.2, and as can be explicitly checked, $\cosh d = -\mathbf{w}_0 \circ \mathbf{w}_1 = -\mathbf{y}_0 \circ \mathbf{y}_1$. The derivation of the formula for $\cosh \ell_1$ is analogous, and we have proved the hyperbolic law of cosines.

For the law of sines we first note that the point of intersection $\mathbf{u}_i$ between S and the geodesic from $\mathbf{v}_i$ in the direction of $\mathbf{x}$ is given by the formula (2), with $\mathbf{v}$ there replaced by $\mathbf{v}_i$, for $i = 0, 1$. From direct calculation and/or Lemma 1.7 we have $\mathbf{v}_i \circ \mathbf{x} = \mathbf{y}_i \circ \mathbf{x}$, whence for each i we have

$$\mathbf{u}_i = \frac{1}{2}\left(1 + \frac{1}{(\mathbf{x} \circ \mathbf{y}_i)^2}\right)\mathbf{x} + \frac{-1}{\mathbf{x} \circ \mathbf{y}_i}\mathbf{y}_0$$

From this we obtain the following formula for the length θ of the horocyclic arc $S \cap P$:

$$\theta = \sqrt{-2(1 + \mathbf{u}_0 \circ \mathbf{u}_1)} = \frac{\sqrt{(\mathbf{x} \circ \mathbf{y}_0)^2 + (\mathbf{x} \circ \mathbf{y}_1)^2 - 2(\mathbf{y}_0 \circ \mathbf{y}_1)(\mathbf{x} \circ \mathbf{y}_0)(\mathbf{x} \circ \mathbf{y}_1)}}{(\mathbf{x} \circ \mathbf{y}_0)(\mathbf{x} \circ \mathbf{y}_1)}$$

Direct computation now establishes this case of the hyperbolic law of sines. □

3. Dimension three: transversals of truncated tetrahedra

We turn now to dimension three, in which hyperplanes are *planes*, ie. two-dimensional totally geodesic copies of $\mathbb{H}^2$. Here we contemplate four different scenarios in which quadruples of objects determine convex regions in $\mathbb{H}^3$:

- **Case TT:** disjoint, non-parallel planes P_1, P_2, P_3, P_4 such that for each i, a single half-space H_i bounded by P_i contains P_j for all $j \neq i$; or
- **Case PT:** for a fixed $k \in \{1, 2, 3\}$, planes $P_1, \dots, P_k$ and horoballs $B_{k+1}, \dots, B_4$, all pairwise disjoint and with the P_i pairwise non-parallel, such that for each $i \leq k$, a single half-space H_i bounded by P_i contains all P_j, $j \neq i$, and $B_{j'}$.

Above, "TT" stands for "Truncated Tetrahedron" and "PT" for "Partially Truncated", referring to the objects that these quadruples determine. In the two subsections below we define these objects and prove trigonometric formulas about their transversal lengths, in the respective cases.

3.1. **The case TT.** The first result of this section helps us define the truncated tetrahedron determined by four planes as in case TT above. We recall the standard fact, proved in eg. [2, Lemma 2.3], that for any collection of three disjoint planes in $\mathbb{H}^3$ there is a unique third plane meeting each of the original three at right angles.

Lemma 3.1. *Suppose P_1, P_2, P_3, P_4 are disjoint, pairwise non-parallel planes in $\mathbb{H}^3$ such that for each i, a single half-space H_i bounded by P_i contains P_j for all $j \neq i$. For any fixed i, let $\widehat{P}_i$ be the plane that intersects P_j at right angles for each $j \neq i$. If P_i does not meet $\widehat{P}_i$ orthogonally then there is a single half-space $\widehat{H}_i$ bounded by $\widehat{P}_i$ such that for all $j \neq i$, $\widehat{H}_i$ contains the shortest geodesic arc from P_j to P_i.*

Proof. Fix $i \in \{1, 2, 3, 4\}$, and let us establish some notation. For each $j \in \{1, 2, 3, 4\}$ let $\mathbf{y}_j$ be an outward unit normal, in the sense described below Lemma 1.5, to the half-space bounded by P_j that contains each other $P_{j'}$. For any $j \neq j'$, $\mathbf{y}_j$ and $\mathbf{y}_{j'}$ are then oppositely-oriented tangent vectors to the hyperbolic geodesic intersecting P_j and $P_{j'}$ perpendicularly, so $\mathbf{y}_j \circ \mathbf{y}_{j'} < 0$ by Lemma 1.8. For each $j \neq i$, let $\mathbf{v}_j$ be the point of intersection between P_j and the geodesic intersecting it and P_i perpendicularly. For each j, Lemma 1.9 gives:

$$\mathbf{v}_j = -\frac{(\mathbf{y}_i \circ \mathbf{y}_j)\,\mathbf{y}_j - \mathbf{y}_i}{\sqrt{(\mathbf{y}_i \circ \mathbf{y}_j)^2 - 1}}. \tag{5}$$

Note that if any such $\mathbf{v}_j$ was contained in $\widehat{P}_i$ then, since both $\widehat{P}_i$ and the shortest geodesic arc from $\mathbf{v}_j$ to P_i intersect P_j at right angles, this entire geodesic arc would be contained in $\widehat{P}_i$. But then P_i would also intersect $\widehat{P}_i$ at right angles, at the other endpoint of this geodesic arc. So because P_i does not intersect $\widehat{P}_i$ at right angles by hypothesis, no such $\mathbf{v}_j$ is contained in $\widehat{P}_i$.

Now fix some $j \neq i$, let $\widehat{H}_i$ be the half-space bounded by $\widehat{P}_i$ that contains $\mathbf{v}_j$, and let $\mathbf{z}_i$ be its outward normal, as described in Lemma 1.5. As noted in the first paragraph above, for any $j' \neq j, i$, $\mathbf{y}_j$ is a tangent vector to the geodesic meeting P_j and $P_{j'}$ perpendicularly. This geodesic lies in $\widehat{P}_i$, so $\mathbf{y}_j$ is a tangent vector to $\widehat{P}_i$ and is therefore orthogonal to $\mathbf{z}_i$. Thus by (5):

$$\mathbf{z}_i \circ \mathbf{v}_j = \frac{\mathbf{z}_i \circ \mathbf{y}_i}{\sqrt{(\mathbf{y}_i \circ \mathbf{y}_j)^2 - 1}}$$

Since $\mathbf{v}_j$ is in the interior of $\widehat{H}_i$, $\mathbf{z}_i \circ \mathbf{v}_j < 0$ by Lemma 1.5. The equation above therefore gives $\mathbf{z}_i \circ \mathbf{y}_i < 0$ as well. But the latter quantity does not depend on j, so this implies that $\mathbf{z}_i \circ \mathbf{v}_{j'} < 0$, and hence that $\mathbf{v}_{j'} \in \widehat{H}_i$ for all $j' \neq i$. The Lemma now follows from the fact that the shortest geodesic arc from any P_j to P_i does not not cross $\widehat{P}_i$, since each intersects P_j at right angles. □

We note that there is a single complementary case to that of Lemma 3.1 in case TT: if P_i intersects $\widehat{P}_i$ at right angles for some i, then the single plane $\widehat{P} \doteq \widehat{P}_i$ intersects all four planes at right angles and thus also equals $\widehat{P}_j$ for each $j \neq i$.

Definition 3.2. Suppose P_1, P_2, P_3, P_4 are disjoint, pairwise non-parallel planes in $\mathbb{H}^3$ such that for each i, a single half-space H_i bounded by P_i contains P_j for all $j \neq i$. For each $i < j \in \{1, 2, 3, 4\}$, denote the shortest arc in $\mathbb{H}^3$ joining P_i to P_j as λ_{ij}.

If $\widehat{P}_i$ as in Lemma 3.1 does not meet P_i orthogonally for any i, the *truncated tetrahedron* determined by the P_i is

$$\Delta = \left(\bigcap_{i=1}^{4} H_i \right) \cap \left(\bigcap_{i=1}^{4} \widehat{H}_i \right),$$

where $\widehat{H}_i$ is the half-space supplied by Lemma 3.1 for each i.

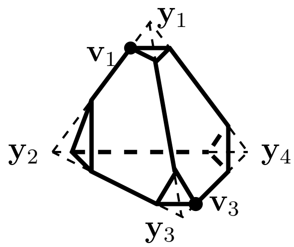

FIGURE 1. A truncated tetrahedron, with missing "vertices" labeled by space-like vectors. Notation as in the proof of Lemma 3.3.

If $\widehat{P}_i$ does meet P_i orthogonally for some i, then taking $\widehat{P} = \widehat{P}_i$ to be the unique plane that intersects each P_i at right angles and renumbering the P_i so that the geodesic containing λ_{13} separates $P_2 \cap \widehat{P}$ from $P_4 \cap \widehat{P}$, we define Δ as a *degenerate truncated tetrahedron* by:

$$\Delta = \widehat{P} \cap \left(\bigcap_{i=1}^{4} H_i \right) \cap h_{12} \cap h_{23} \cap h_{34} \cap h_{14},$$

where h_{12} is the half-plane in $\widehat{P}$ that is bounded by the geodesic containing λ_{12} and contains $P_3 \cap \widehat{P}$ and $P_4 \cap \widehat{P}$; and so on.

For each $i < j \in \{1, 2, 3, 4\}$, call λ_{ij} an *internal edge* of Δ. The internal edge *opposite* λ_{ij} is λ_{kl}, where $k < l \in \{1, 2, 3, 4\} - \{i, j\}$. For each i, the *internal face opposite* P_i is the right-angled hexagon in $\Delta \cap \widehat{P}_i$ bounded by the internal edges λ_{jk}, for each pair $j < k \in \{1, 2, 3, 4\} - \{i\}$, and arcs of the P_j, $j \neq i$. The non-internal faces and edges of Δ are *external.* Each of these is entirely contained in P_i for some i.

The *transversal* of Δ joining an internal edge λ_{ij} to its opposite λ_{kl} is the shortest geodesic arc with one endpoint on each edge; or if these edges intersect, it is their point of intersection.

Note that if for some $i < j$, λ_{ij} intersects its opposite edge λ_{kl}, then Δ is degenerate since the plane containing both λ_{ij} and λ_{kl} intersects all four P_i orthogonally. Conversely, if Δ is degenerate then it is a right-angled octagon in $\widehat{P}$, and with the P_i numbered as in this case of Definition 3.2, the opposite edges λ_{13} and λ_{24} do intersect.

In the non-degenerate case, each internal face of Δ is of the form $\Delta \cap \widehat{P}_i$ for some i, and each edge λ_{ij} is the intersection of the internal faces contained in $\widehat{P}_k$ and $\widehat{P}_l$ for $k < l \in \{1, 2, 3, 4\} - \{i, j\}$. In this case, Δ is homeomorphic to the complement in a tetrahedron of the union of small regular neighborhoods of the vertices; see Figure 1.

The main results of this section record some observations about the lengths of transversals of truncated tetrahedra. Before embarking on this we record the following basic geometric fact.

Lemma 3.3. *Suppose P_1, P_2, P_3, P_4 are disjoint, pairwise non-parallel planes in $\mathbb{H}^3$ such that for each i, a single half-space H_i bounded by P_i contains P_j for all $j \neq i$; and for any $i < j \in \{1, 2, 3, 4\}$, let $\tilde{\lambda}_{ij}$ be the geodesic intersecting P_i and P_j at right angles. Fixing such an $i < j$ and $k < l \in \{1, 2, 3, 4\} - \{i, j\}$, and fixing parametrizations $\tilde{\lambda}_{ij}(s)$ and $\tilde{\lambda}_{kl}(t)$ by arclength, the function $D(s, t)$ that records the hyperbolic cosine of the distance from $\tilde{\lambda}_{ij}(s)$ to $\tilde{\lambda}_{kl}(t)$ has a unique critical point in $\mathbb{R}^2$, at which it attains an absolute minimum. The absolute minimum value depends only on the pairwise distances between the planes.*

All assertions above but the final one follow from standard geometric facts in a straightforward way, but we will prove them all here using the general perspective taken in this note.

Proof. Similarly to the proof of Lemma 3.1, for each $i \in \{1,2,3,4\}$ let $\mathbf{y}_i$ be an outward unit normal, in the sense described below Lemma 1.5, to the half-space H_i bounded by P_i that contains each other P_j. For any $j \neq i$, $\mathbf{y}_i$ and $\mathbf{y}_j$ are then oppositely-oriented tangent vectors to the hyperbolic geodesic intersecting P_i and P_j perpendicularly, so $\mathbf{y}_i \circ \mathbf{y}_j < 0$ by Lemma 1.8. Furthermore, by this result the hyperbolic cosine of the distance from P_i to P_j, which we will here denote L_{ij}, satisfies $L_{ij} = -\mathbf{y}_i \circ \mathbf{y}_j$ for each $i < j \in \{1,2,3,4\}$.

Since the P_i are labeled arbitrarily, we may take $(i,j,k,l) = (1,2,3,4)$ without loss of generality. We will also prove the Lemma's conclusion for particular parametrizations of $\tilde{\lambda}_{12}$ and $\tilde{\lambda}_{34}$ below. This will imply the general result, since for any other parametrization the resulting $D(s,t)$ will be obtained from this one by precomposing with a translation of $\mathbb{R}^2$ and a map of the form $(s,t) \mapsto (\pm s, \pm t)$.

Let $\mathbf{v}_1$ and $\mathbf{v}_3$ be the points of intersection $\tilde{\lambda}_{12} \cap P_1$ and $\tilde{\lambda}_{34} \cap P_3$, respectively. Then $\tilde{\lambda}_{12}$ is parametrized by arclength as $\tilde{\lambda}_{12}(s) = \cosh s\,\mathbf{v}_1 - \sinh s\,\mathbf{y}_1$ starting at $\mathbf{v}_1$ and running into H_1, since $\mathbf{y}_1$ is outward-pointing from H_1, and likewise $\tilde{\lambda}_{34}(t) = \cosh t\,\mathbf{v}_3 - \sinh t\,\mathbf{y}_3$ starts at $\mathbf{v}_3$ and runs into H_3. $D(s,t)$ described above thus satisfies

$$\begin{aligned} D(s,t) &= -(\cosh s\,\mathbf{v}_1 - \sinh s\,\mathbf{y}_1) \circ (\cosh t\,\mathbf{v}_3 - \sinh t\,\mathbf{y}_3) \\ &= -\cosh s \cosh t\,(\mathbf{v}_1 \circ \mathbf{v}_3) + \cosh s \sinh t\,(\mathbf{v}_1 \circ \mathbf{y}_3) \\ &\quad + \sinh s \cosh t\,(\mathbf{y}_1 \circ \mathbf{v}_3) - \sinh s \sinh t\,(\mathbf{y}_1 \circ \mathbf{y}_3) \end{aligned}$$

We claim first that $D(s,t) \to \infty$ as s and t escape compact sets. To see this we write:

$$\frac{D(s,t)}{\cosh s \cosh t} = -(\mathbf{v}_1 \circ \mathbf{v}_3) + \tanh t\,(\mathbf{v}_1 \circ \mathbf{y}_3) + \tanh s\,(\mathbf{y}_1 \circ \mathbf{v}_3) - \tanh s \tanh t\,(\mathbf{y}_1 \circ \mathbf{y}_3), \tag{6}$$

and record each inner product above in terms of the distances L_{ij} from P_i to P_j by substituting the formulas for $\mathbf{v}_1$ and $\mathbf{v}_3$ from Lemma 1.9.

$$\begin{aligned} &\mathbf{v}_1 \circ \mathbf{v}_3 = -\frac{L_{12}L_{13}L_{34} + L_{23}L_{34} + L_{12}L_{14} + L_{24}}{\sqrt{(L_{12}^2 - 1)(L_{34}^2 - 1)}}, \\ &\mathbf{v}_1 \circ \mathbf{y}_3 = -\frac{L_{12}L_{13} + L_{23}}{\sqrt{L_{12}^2 - 1}}, \quad \mathbf{y}_1 \circ \mathbf{v}_3 = -\frac{L_{13}L_{34} + L_{14}}{\sqrt{L_{34}^2 - 1}}, \quad \mathbf{y}_1 \circ \mathbf{y}_3 = -L_{13} \end{aligned} \tag{7}$$

For large enough values of t, values of the ratio (6) can be made arbitrarily close to $-(\mathbf{v}_1 \circ \mathbf{v}_3) + (\mathbf{v}_1 \circ \mathbf{y}_3) + \tanh s(\mathbf{y}_1 \circ \mathbf{v}_3) - \tanh s(\mathbf{y}_1 \circ \mathbf{y}_3)$. Substituting from (7), we write this as:

$$\begin{aligned} &\left[\frac{L_{34}}{\sqrt{L_{34}^2 - 1}} - 1\right]\left[L_{13}\left(\frac{L_{12}}{\sqrt{L_{12}^2 - 1}} - \tanh s\right) + \frac{L_{23}}{\sqrt{L_{12}^2 - 1}}\right] \\ &\qquad\qquad + \frac{L_{14}}{\sqrt{L_{34}^2 - 1}}\left[\frac{L_{12}}{\sqrt{L_{12}^2 - 1}} - \tanh s\right] + \frac{L_{24}}{\sqrt{(L_{12}^2 - 1)(L_{34}^2 - 1)}} \end{aligned}$$

Since $\tanh s < 1$ for all s, this sum exceeds a fixed positive bound regardless of the value of s.

Now as t decreases without bound, the ratio (6) approaches $-(\mathbf{v}_1 \circ \mathbf{v}_3) - (\mathbf{v}_1 \circ \mathbf{y}_3) + \tanh s(\mathbf{y}_1 \circ \mathbf{v}_3) + \tanh s(\mathbf{y}_1 \circ \mathbf{y}_3)$. This is again seen to have a positive lower bound, upon writing it as:

$$\left[\frac{L_{13}L_{34} + L_{14}}{\sqrt{L_{34}^2 - 1}} + L_{13}\right]\left[\frac{L_{12}}{\sqrt{L_{12}^2 - 1}} - \tanh s\right] + \frac{L_{23}L_{34} + L_{24}}{\sqrt{(L_{12}^2 - 1)(L_{34}^2 - 1)}} + \frac{L_{23}}{\sqrt{L_{12}^2 - 1}}$$

We thus have a fixed positive lower bound for the ratio $D(s,t)/(\cosh s \cosh t)$, for large enough values of $|t|$. It follows that $D(s,t) \to \infty$ with $|t|$, regardless of the value of s. A parallel argument to the above (or again using the fact that the numbering of the P_i is arbitrary) shows that the same is true with the roles of s and t reversed, proving the claim.

The claim implies that $D(s,t)$ does attain a minimum on $\mathbb{R}^2$, necessarily occurring at a critical point. Our next goal is to classify critical points by setting the gradient $\nabla D(s,t) = \mathbf{0}$ equal to 0 and solving for (s,t). Computing partials and dividing by $\cosh s \cosh t$ yields:

$$\frac{\partial D/\partial s}{\cosh s \cosh t} = -\tanh s\,(\mathbf{v}_1 \circ \mathbf{v}_3) + \tanh s \tanh t\,(\mathbf{v}_1 \circ \mathbf{y}_3) + \mathbf{y}_1 \circ \mathbf{v}_3 - \tanh t\,(\mathbf{y}_1 \circ \mathbf{y}_3)$$
$$\frac{\partial D/\partial t}{\cosh s \cosh t} = -\tanh t\,(\mathbf{v}_1 \circ \mathbf{v}_3) + (\mathbf{v}_1 \circ \mathbf{y}_3) + \tanh s \tanh t\,(\mathbf{y}_1 \circ \mathbf{v}_3) - \tanh s\,(\mathbf{y}_1 \circ \mathbf{y}_3)$$

Setting these two equations equal to 0 simultaneously, then solving the first for $\tanh s$ in terms of $\tanh t$ and substituting into the second, yields the quadratic $a(\tanh t)^2 + b\tanh t + a = 0$ for

$$a = (\mathbf{y}_1 \circ \mathbf{y}_3)(\mathbf{y}_1 \circ \mathbf{v}_3) - (\mathbf{v}_1 \circ \mathbf{y}_3)(\mathbf{v}_1 \circ \mathbf{v}_3) \ \text{ and } \ b = (\mathbf{v}_1 \circ \mathbf{v}_3)^2 + (\mathbf{v}_1 \circ \mathbf{y}_3)^2 - (\mathbf{y}_1 \circ \mathbf{v}_3)^2 - (\mathbf{y}_1 \circ \mathbf{y}_3)^2.$$

Direct computation shows that $a < 0$ and that

$$\begin{aligned} b + 2a &= (\mathbf{v}_1 \circ \mathbf{v}_3 - \mathbf{v}_1 \circ \mathbf{y}_3)^2 - (\mathbf{y}_1 \circ \mathbf{v}_3 - \mathbf{y}_1 \circ \mathbf{y}_3)^2 \\ &= (\mathbf{v}_1 \circ \mathbf{v}_3 - \mathbf{v}_1 \circ \mathbf{y}_3 - \mathbf{y}_1 \circ \mathbf{v}_3 + \mathbf{y}_1 \circ \mathbf{y}_3)(\mathbf{v}_1 \circ \mathbf{v}_3 - \mathbf{v}_1 \circ \mathbf{y}_3 + \mathbf{y}_1 \circ \mathbf{v}_3 - \mathbf{y}_1 \circ \mathbf{y}_3) \\ &> 0, \ \text{being a product of negative numbers.} \end{aligned}$$

(The relevant computations here are very similar to those above.) It therefore follows that the roots of the quadratic above in $\tanh t$ are positive and their product is one, with one root larger than one and one smaller. Since $\tanh t$ takes only values less than 1, only the smaller root corresponds to a critical point. Therefore $D(s,t)$ has a unique critical point, which must correspond to its global minimum.

The minimum of $D(s,t)$ thus occurs at a value (s,t) that is given in terms of the L_{ij} by the quadratic formula. Since the values of $D(s,t)$ depend only on s, t, and the L_{ij}, it follows that the minimum value itself depends only on the L_{ij}. □

Lemma 3.4. *Suppose P_1, P_2, P_3, P_4 are disjoint, pairwise non-parallel planes in $\mathbb{H}^3$ such that for each i, a single half-space H_i bounded by P_i contains P_j for all $j \neq i$, and let Δ be the truncated tetrahedron determined by the P_i as in Definition 1. For any fixed $i < j \in \{1,2,3,4\}$ and $k < l \in \{1,2,3,4\} - \{i,j\}$, the length of the transversal of Δ joining λ_{ij} to λ_{kl} equals the minimum distance between the geodesics $\tilde{\lambda}_{ij}$ and $\tilde{\lambda}_{kl}$ respectively containing them. It depends only on the lengths of the internal edges of Δ, and written as $T(x,y;a,b,c,d)$, where $x = \cosh \ell(\lambda_{ij})$, $y = \cosh \ell(\lambda_{kl})$, and a, b, c, d are hyperbolic cosines of the other four internal edge lengths of Δ, it is symmetric in a, b, c and d, and strictly increasing in each of them, for any fixed $x, y > 1$.*

Proof. We note that $\tilde{\lambda}_{ij}$ and $\tilde{\lambda}_{kl}$ here are the same-named geodesics from Lemma 3.3. Given this, the claim that the transversal length is the minimum distance between them implies that $T(x,y;a,b,c,d)$ equals the minimum value of $D(s,t)$ from that result. The fact that it depends only on the internal edge lengths will thus follow directly from the conclusion of Lemma 3.3—since the internal edge lengths are the pairwise distances between planes—and the symmetry will follow from the fact that the labeling of the P_i, and hence also the L_{ij}, is arbitrary.

To prove the claim we establish first that $\tilde{\lambda}_{ij}$ intersects $\tilde{\lambda}_{kl}$, if at all, in $\lambda_{ij} \cap \lambda_{kl}$. The two components of $\tilde{\lambda}_{ij} - \lambda_{ij}$ are contained in the half-spaces bounded by P_i and P_j opposite H_i and H_j, respectively; the analog is true for the components of $\tilde{\lambda}_{kl} - \lambda_{kl}$; and this implies that each component of $\tilde{\lambda}_{ij} - \lambda_{ij}$ is entirely disjoint from $\tilde{\lambda}_{kl}$. Therefore if $\tilde{\lambda}_{ij}$ intersects $\tilde{\lambda}_{kl}$, it does so in $\lambda_{ij} \cap \lambda_{kl}$. In this case the transversal length and the minimum of $D(s,t)$ both equal 0.

We now suppose that $\tilde{\lambda}_{ij}$ does not intersect $\tilde{\lambda}_{kl}$, and let γ be the unique shortest geodesic arc from $\tilde{\lambda}_{ij}$ to $\tilde{\lambda}_{kl}$. That is, γ joins $\tilde{\lambda}_{ij}(s_0)$ to $\tilde{\lambda}_{kl}(t_0)$, where the unique minimum of $D(s,t)$ occurs at (s_0,t_0). By a standard argument, γ intersects each of $\tilde{\lambda}_{ij}$ and $\tilde{\lambda}_{kl}$ at a right angle (if not, one could reduce its length by moving an endpoint at a non-right angle of intersection in the direction of the smaller angle). Because $\tilde{\lambda}_{ij}$ intersects each of P_i and P_j at right angles, any geodesic arc with an endpoint on a component of $\tilde{\lambda}_{ij} - \lambda_{ij}$ is entirely contained in the half-space bounded by P_i or P_j containing that component. Such an arc is therefore disjoint from $\tilde{\lambda}_{kl}$, and it follows that γ has its endpoints on λ_{ij} and λ_{kl}. This proves the claim.

It remains to show that $T(x, y; a, b, c, d)$ is increasing in each of its last four variables. For this, as in the proof of Lemma 3.3 we take $i = 1$, $j = 2$, $k = 3$, and $l = 4$ after renumbering the P_i if necessary. The value $T(x, y; a, b, c, d)$ is then the minimum of the function $D(s, t)$ defined there. Although it is not explicit in the notation, the function D itself depends on the values $x = L_{12}$, $y = L_{34}$, $a = L_{13}$, $b = L_{14}$, $c = L_{23}$, and $d = L_{24}$. To exhibit this, we plug the values from (7) into the formula of (6), yielding:

$$\frac{D(s,t)}{\cosh s \cosh t} = \frac{L_{12}L_{13}L_{34} + L_{12}L_{14} + L_{23}L_{34} + L_{24}}{\sqrt{(L_{12}^2 - 1)\ (L_{34}^2 - 1)}} \\ - \tanh t \frac{L_{12}L_{13} + L_{23}}{\sqrt{L_{12}^2 - 1}} - \tanh s \frac{L_{13}L_{34} + L_{14}}{\sqrt{L_{34}^2 - 1}} + \tanh s \tanh t\ L_{13}.$$

The value $d = L_{24}$ appears only once in this formula, in the numerator of the first summand above, with a positive sign. It is thus clear that for any fixed s and t, the value of $D(s, t)$ increases with d (regarding the other L_{ij} as fixed). Its absolute minimum, and hence the transversal length $T(x, y; a, b, c, d)$, therefore also increases with d. Since $T(x, y; a, b, c, d)$ is symmetric in a, b, c, and d, the same then holds for a, b and c. □

Proposition 3.5. *Suppose P_1, P_2, P_3, P_4 are disjoint, pairwise non-parallel planes in $\mathbb{H}^3$ such that for each i, a single half-space H_i bounded by P_i contains P_j for all $j \neq i$, and let Δ be the truncated tetrahedron determined by the P_i as in Definition 1. For any fixed $i < j \in \{1, 2, 3, 4\}$ and $k < l \in \{1, 2, 3, 4\} - \{i, j\}$, let $T(x, y; a, b, c, d)$ record the length of the transversal of Δ joining λ_{ij} to λ_{kl} as in Lemma 3.2, where $x = \cosh \ell(\lambda_{ij})$, $y = \cosh \ell(\lambda_{kl})$, and a, b, c, d are hyperbolic cosines of the other four internal edge lengths of Δ. For some fixed $L > 1$, if each of a, b, c, and d is at least L, then*

$$\cosh T(x, y; a, b, c, d) \geq \frac{2L}{\sqrt{(x-1)(y-1)}},$$

with equality if and only if $a = b = c = d = L$.

The Proposition's proof rests on our ability to explicitly locate the critical point of the function $D(s, t)$ from Lemma 3.3, and hence explictly compute values of T, in highly symmetric situations.

Proof. We re-record the formulas of (7) in the special case that $L_{13} = L_{14} = L_{23} = L_{24} = L$, and $L_{12} = x$, $L_{34} = y$:

$$\mathbf{v}_1 \circ \mathbf{v}_3 = -L\sqrt{\frac{(x+1)(y+1)}{(x-1)(y-1)}}, \quad \mathbf{v}_1 \circ \mathbf{y}_3 = -L\sqrt{\frac{x+1}{x-1}}, \quad \mathbf{y}_1 \circ \mathbf{v}_3 = -L\sqrt{\frac{y+1}{y-1}}, \quad \mathbf{y}_1 \circ \mathbf{y}_3 = -L$$

We claim that if $a = b = c = d = L$ then the minimum of $D(s, t)$ occurs at (s_0, t_0) where $s_0 = \ell(\lambda_{12})/2$ and $t_0 = \ell(\lambda_{34})/2$, yielding $\cosh s_0 = \sqrt{\frac{1}{2}(x+1)}$ and $\cosh t_0 = \sqrt{\frac{1}{2}(y+1)}$. This can be proved by substituting directly into the formulas for $\partial D/\partial s$ and $\partial D/\partial t$ from the proof of Lemma 3.3, showing that this (s_0, t_0) is the unique critical point of D. Plugging it into the formula for D from the Lemma yields:

$$\begin{aligned} D(s_0,t_0) &= -\cosh s_0 \cosh t_0\,(\mathbf{v}_1\circ\mathbf{v}_3) + \cosh s_0 \sinh t_0\,(\mathbf{v}_1\circ\mathbf{y}_3) \\ &\qquad + \sinh s_0 \cosh t_0\,(\mathbf{y}_1\circ\mathbf{v}_3) - \sinh s_0 \sinh t_0\,(\mathbf{y}_1\circ\mathbf{y}_3) \\ &= \frac{L}{2}\left[\frac{(x+1)(y+1)}{\sqrt{(x-1)(y-1)}} - \frac{(x+1)\sqrt{y-1}}{\sqrt{x-1}} - \frac{\sqrt{x-1}(y+1)}{\sqrt{y-1}} + \sqrt{(x-1)(y-1)}\right] \end{aligned}$$

When simplified, this yields the formula of the Proposition statement. It now follows from Lemma 3.4 that this bounds the value of $\cosh T(x,y;a,b,c,d)$ below when a, b, c, and d are all at least L, and that equality holds if and only if $a=b=c=d=L$. $\square$

3.2. **The case PT.** We now change the set-up slightly by replacing the plane P_4 with a horoball B disjoint from P_1, P_2, and P_3, and such that for each $i\in\{1,2,3\}$, P_i bounds a half-space H_i containing B and the other two hyperplanes.

If the ideal point of B does not lie in the mutual perpendicular $\widehat{P}$ to P_1, P_2, and P_3, then we define the *partially truncated tetrahedron determined by B and the P_i* to be the intersection of the H_i, $i=1,2,3$, with the half-space H bounded by $\widehat{P}$ that contains the ideal point of B, and three half-spaces $\widehat{H}_i$, $i=1,2,3$. For each such i, $\widehat{H}_i$ is bounded by the mutual perpendicular $\widehat{P}_i$ to the other two planes P_j, P_k that contains the ideal point of B and hence meets B perpendicularly. $\widehat{H}_i$ is the half-space bounded by P_i that contains the shortest geodesic arc from B to P_i.

Definition 3.6. Taking a, b and c to be the distances from P_1 to P_2, P_2 to P_3 and P_3 to P_1, respectively, and h_i to be the distance from P_i to B, for each $i\in\{1,2,3\}$, denote the partially truncated tetrahedron constructed as above by $T(h_1,h_2,h_3,a,b,c)$.

Proposition 3.7. *For $T \doteq T(h_1,h_2,h_3,a,b,c)$ as in Definition 3.6, if $h_i=h$ for each i, and $a=b=c=\ell_1$, for fixed h and $\ell_1>0$, then the distance D between the edge of T joining P_1 to B and the edge joining P_2 to P_3 satisfies*

$$\cosh D = 2\sqrt{1+\frac{\cosh\ell_1\sqrt{2}}{\sqrt{\cosh\ell_1-1}}}.$$

Proof. Let $\mathbf{x}\in\mathbb{R}^{1,3}$ be the positive light-like vector that determines the horoball B, and for $i=1,2,3$ let $\mathbf{y}_i$ be a unit space-like vector in $\mathbb{R}^3$ normal to P_i and such that $\mathbf{x}\circ\mathbf{y}_i<0$ for each i. Then also, $\mathbf{y}_i\circ\mathbf{y}_j<0$ for $j\neq i$ by hypothesis.

By the proof of Lemma 1.7, the geodesic ray $\gamma(t)$ from the closest point $\mathbf{v}_1$ on P_1 in the direction of $\mathbf{x}$ satisfies

$$\gamma(t) = e^{-t}\mathbf{v}_1 - \frac{\sinh t}{\mathbf{x}\circ\mathbf{v}_1}\mathbf{x} = \frac{\cosh t}{-\mathbf{x}\circ\mathbf{y}_1}\mathbf{x} + e^{-t}\mathbf{y}_1.$$

We wish to minimize the distance from $\gamma(t)$ to the geodesic from P_2 to P_3. Taking $L_1=\cosh\ell_1$ we recall that this is parametrized as

$$\lambda_{23}(s) = \cosh s\,\mathbf{z}_2 + \sinh s\,\mathbf{y}_2, \quad \text{for} \quad \mathbf{z}_2 = \frac{(\mathbf{y}_2\circ\mathbf{y}_3)\mathbf{y}_2-\mathbf{y}_3}{\sqrt{(\mathbf{y}_2\circ\mathbf{y}_3)^2-1}} = \frac{L_1\,\mathbf{y}_2-\mathbf{y}_3}{\sqrt{L_1^2-1}}$$

Due to the symmetry of the situation, the closest point of λ_{23} to $\gamma(t)$ is its midpoint s_0, which satisfies $\cosh s_0 = \sqrt{\frac{L_1+1}{2}}$ and $\sinh s_0 = \sqrt{\frac{L_1-1}{2}}$. Plugging this in gives $\gamma(s_0) = (-\mathbf{y}_2 -$

$\mathbf{y}_3)/\sqrt{2(L_1-1)}$. We thus are looking to minimize

$$\begin{aligned}\gamma(t)\circ\lambda_{23}(s_0) &= \frac{\mathbf{x}\circ\mathbf{y}_2+\mathbf{x}\circ\mathbf{y}_3}{\mathbf{x}\circ\mathbf{y}_1}\cosh t-\frac{\mathbf{y}_1\circ\mathbf{y}_2+\mathbf{y}_1\circ\mathbf{y}_3}{\sqrt{2(L_1-1)}}\,e^{-t}\\ &= 2\cosh t+\frac{2L_1}{\sqrt{2(L_1-1)}}\,e^{-t}=e^t+\left(1+\frac{L_1\sqrt{2}}{\sqrt{L_1-1}}\right)e^{-t}\end{aligned}$$

Setting a derivative equal to 0 yields

$$e^{2t}=1+\frac{L_1\sqrt{2}}{\sqrt{L_1-1}}\qquad\Rightarrow\qquad e^t=\sqrt{1+\frac{L_1\sqrt{2}}{\sqrt{L_1-1}}}$$

Plugging this into $\gamma(t)\circ\lambda_{23}(s_0)$ yields the formula given above. □

Arguing as for Lemma 3.4, we obtain:

Lemma 3.8. *For $T\doteq T(h_1,h_2,h_3,a,b,c)$ as in Definition 3.6, if $h_i\geq h$ for each i, and a, b, and c are all at least ℓ_1, for fixed h and $\ell_1>0$, then the distance D between the edge of T joining P_1 to B and the edge joining P_2 to P_3 is at least D from Proposition 3.7.*

We now replace another plane by a horoball. That is, consider a collection P_1, P_2, B_1, B_2 of mutually disjoint planes (the P_i) and horoballs (the B_i) such that for each $i\in\{1,2\}$, B_1, B_2, and P_{3-i} are contained in a single complementary component of P_i. As in the previous case, there is a partially truncated tetrahedron determined by the P_i, $i=1,2$, and the four planes orthogonal to triples of the four vertex objects.

Definition 3.9. Taking ℓ be the distance from P_1 to P_2, d the distance from B_1 to B_2, and h_{ij} the distance from P_i to B_j, for $i,j\in\{1,2\}$, denote the partially truncated tetrahedron constructed above by $T(d,h_{11},h_{12},h_{21},h_{22},\ell)$.

Proposition 3.10. *For $T\doteq T(d,h_{11},h_{12},h_{21},h_{22},\ell)$ as in Definition 3.9, if $h_{ij}=h$ for each $i,j\in\{1,2\}$, then the distance D from the edge of T joining P_1 to B_1 to the edge joining P_2 to B_2 satisfies*

$$\cosh D=1+\frac{e^d}{e^{2h}}+\sqrt{\frac{e^d}{e^{2h}}}\sqrt{2+2\cosh\ell+\frac{e^d}{e^{2h}}}.$$

Proof. For $i=1,2$, let $\mathbf{x}\in\mathbb{R}^{1,3}$ be the positive light-like vector that determines the horoball B_i, and let $\mathbf{y}_i$ be a unit space-like vector in $\mathbb{R}^3$ normal to P_i and such that $\mathbf{x}_i\circ\mathbf{y}_j<0$ for each i. For each i, the geodesic ray γ_i from the closest point $\mathbf{v}_i$ of P_i in the direction of $\mathbf{x}_i$ is given by

$$\gamma_i(t)=\frac{\cosh t}{-\mathbf{x}_i\circ\mathbf{y}_i}\mathbf{x}_i+e^{-t}\mathbf{y}_i.$$

Let λ be the mutual perpendicular to the geodesic joining the ideal point of B_1 to that of B_2 and the geodesic containing the shortest arc between P_1 and P_2. The π-rotation around λ exchanges B_i with B_{3-i} and P_i with P_{3-i}, for $i=1,2$. Thus it takes $\gamma_i(s)$ to $\gamma_{3-i}(s)$ for $i=1,2$. It follows that the shortest distance between $\gamma_1(s)$ and $\gamma_2(t)$ is realized at some $s=t$.

To identify this t, we set $\frac{d}{dt}\left[-\gamma_1(t)\circ\gamma_2(t)\right]$ equal to 0. After simplification, this yields:

$$e^{2t}=\sqrt{\frac{2+2\cosh\ell_1+e^d/e^{2h}}{e^d/e^{2h}}}$$

Plugging this back into $-\gamma_1(t)\circ\gamma_2(t)$ yields the result. □

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