RESIDUAL FINITENESS GROWTHS OF CLOSED HYPERBOLIC MANIFOLDS IN NON-COMPACT RIGHT-ANGLED REFLECTION ORBIFOLDS

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Abstract. We bound the geodesic residual finiteness growths of closed hyperbolic manifolds that immerse totally geodesically in non-compact right-angled reflection orbifolds, extending work of the fourth author from the compact case. We describe a rich class of arithmetic examples to which this result applies, and apply the theorem to give effective bounds on the geodesic residual finiteness growths of members of this family.

This paper contributes a few observations to the project of *quantifying residual finiteness growth*. Our first main result, Theorem 1.6, extends those of [16], which established explicit linear bounds on the geodesic residual finiteness growth of fundamental groups of closed hyperbolic manifolds that admit totally geodesic immersions to a compact right-angled reflection orbifolds. Our results still concern closed manifolds, but we allow non-compact right-angled reflection orbifolds of finite volume. Our bound depends on a choice of "embedded" horoballs.

Definition 1. For a polyhedron P and an ideal vertex v of P, we will say a horoball centered at v is *embedded* in P if it does not intersect the interior of any side of P that is not incident on v.

Here and below, the term "side" of a polyhedron P refers specifically to a codimension-one face of P, following Ratcliffe (see [19], p. 198 and Theorem 6.3.1).

Theorem 1.6. *For $n \geq 2$, let P be a right-angled polyhedron in $\mathbb{H}^{n+1}$ with finite volume and at least one ideal vertex, let Γ_P be the group generated by reflections in the sides of P, and let $\mathcal{B}$ be a collection of horoballs, one for each ideal vertex of P, that are each embedded in the sense of Definition 1 and pairwise non-overlapping. For a closed hyperbolic m-manifold M, $m \leq n$, that admits a totally geodesic immersion to $\mathcal{O}_P \doteq \mathbb{H}^{n+1}/\Gamma_P$, and any $\alpha \in \pi_1 M - \{1\}$, there exists a subgroup H' of $\pi_1 M$ such that $\alpha \notin H'$, and the index of H' is bounded above by*

$$\frac{2v_n(1)}{V_{R+h_{\max}}} \sinh^n\left(R + d_{R+h_{\max}}\right) \ell(\alpha),$$

where $v_n(1)$ is the (Euclidean) volume of the n-dimensional Euclidean unit ball and:

- *$\ell(\alpha)$ is the length of the unique geodesic representative of α;*
- *$R = \ln(\sqrt{n+1} + \sqrt{n})$;*
- *$h_{\max} = \ln(\cosh r_{\max})$, where $r_{\max}$ is the radius of the largest embedded ball in M; and*
- *$d_{R+h_{\max}}$ and $V_{R+h_{\max}}$ are the diameter and volume, respectively, of the $(R + h_{\max})$-neighborhood in P of $\overline{P - \bigcup\{B \in \mathcal{B}\}}$.*

The bound above is the natural extension of those of Theorems 3.3 and 4.3 of [16], with the role of the polyhedron P there played here by a "compact core": the $(R + h_{\max})$-neighborhood of $\overline{P - \bigcup\{B \in \mathcal{B}\}}$. Because $h_{\max}$ appears here, the resulting bound depends not only on P but also on M, unlike in [16]. This reflects the fact that we use the radius of the largest embedded ball in M to control its interaction with the thin part of P, where the techniques of [16] break down.

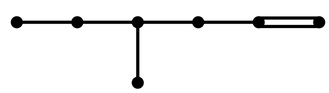
Theorem 1.6 applies to a significantly larger class of examples than [16]. In particular, while compact right-angled polyhedra exist in $\mathbb{H}^n$ only for $n \leq 4$, there are finite-volume examples up to dimension at least eight (see Potyagailo–Vinberg [17]). And many closed manifolds immerse in right-angled reflection orbifolds of higher dimension. Our second main result exhibits a rich class of such manifolds.

Theorem 2.3. *Let Γ be an arithmetic lattice that is commensurable with $\mathrm{SO}(q, \mathbb{Z})$ for some $\mathbb{Q}$–defined bilinear form q. Then there exists an absolute constant C_1 and a constant D that depends only on the commensurability class of Γ such that Γ has a subgroup Δ of index at most $C_1 D \mathrm{covol}(\Gamma)$ and an injective homomorphism from Δ to a subgroup of $\mathrm{SO}(6, 1; \mathbb{Z})$ that stabilizes a time-like subspace of $\mathbb{R}^{6,1}$.*

Thus for an arithmetic hyperbolic manifold $M = \mathbb{H}^3/\Gamma$, where Γ is such a lattice, there is a cover $\tilde{M} \to M$ of degree at most $C_1 D \mathrm{vol}(M)$ with a totally geodesic immersion to $\mathbb{H}^6/\Gamma_P$, for a right-angled polyhedron P.

This result uses the strategy exploited by Agol–Long–Reid in [1]. The main idea is to exhibit the quadratic form q associated to Γ as a sub-form of a seven-dimensional form which is conjugate over $\mathbb{Q}$ to the standard form of signature $(6, 1)$. It is a standard fact that Γ is cocompact if q is anisotropic over $\mathbb{Q}$; i.e. if $q(v, v) \neq 0$ for all $v \in \mathbb{Q}^4$. There are infinitely many commensurability classes of such manifolds.

The Theorem's second assertion follows from the fact that $\mathrm{SO}(6, 1; \mathbb{Z})$ is the group generated by reflections in the sides of a simplex σ in $\mathbb{H}^6$ that has one ideal vertex, which is itself a fundamental domain for the symmetry group of a right-angled ideal polyhedron P in $\mathbb{H}^6$. The Coxeter diagram for the reflection group in the sides of σ is pictured below; compare [1, Fig. 1] and [19, Fig. 7.3.4].



Each face of the right-angled polyhedron P is a union of G-translates of the side of σ corresponding to the diagram's rightmost vertex, where G is the finite group generated by reflections in all other sides of σ. See Lemma 3.4 of [1] and its proof.

In Section 3 we illustrate the effectiveness of our methods by collecting enough geometric data on the simplex σ above to give an explicit bound on the index of a subgroup of $\pi_1 M$ excluding an element α, in the setting of Theorem 2.3, which depends only on the defining quadratic form q and the volume of M.

Corollary 3.3. *Suppose $M = \mathbb{H}^3/\Gamma$ is a closed arithmetic hyperbolic 3-manifold such that Γ is commensurable with $\mathrm{SO}(q, \mathbb{Z})$ for some $\mathbb{Q}$-defined form q. For any $\alpha \in \Gamma - \{1\}$, there exists a subgroup H' of $\pi_1 M$ such that $\alpha \notin H'$, and the index of H' is bounded above by*

$$2^7 3^4 5 \cdot C_1 x^{21} \mathrm{vol}(M) \cdot \frac{v_5(1)}{V_0} \sinh^5\left(2(2R + d_{\max} + \log p^{-1}(\mathrm{vol}(M)))\right) \ell(\alpha),$$

where $v_5(1) = 8\pi^2/15$ *and:*

- $\ell(\alpha)$ *is the length of the unique geodesic representative of* α*;*
- $R = \ln(\sqrt{6}+\sqrt{7})$*;*
- $C_1 = 2^{2\log(3)/\log(3/2)}/\log(3/2)$*, see Lemma 2.1;*
- $x = x(q)$ *is as described in the formula (5);*
- $V_0 = \frac{2^{2.5}\pi^3 - 3^4}{2^{2.5}\cdot 5\cdot 3} \approx 1.112$ *and* $d_{\max} = \cosh^{-1}(\sqrt{3})$*, see Corollary 3.2; and*
- $p(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x - \frac{8}{15}$.

Our work compares naturally to that of Bou-Rabee–Hagen–Patel [8], who showed that any subgroup G of a right-angled Artin group has linear *residual finiteness growth*, which is computed with respect to the word metric determined by a fixed generating set for G. (Its linearity does not depend on the choice of generating set, cf. [8, Lemma 2.1].) Because hyperbolic 3-manifold groups are virtually special [2], this yields linear residual finiteness growth for *every* finite-volume hyperbolic 3-manifold. This in turn implies linear *geodesic* residual finiteness growth for every closed hyperbolic 3-manifold, see [16, Lemma 6.1].

The virtually special machine currently offers no effective method for determining, given an arbitrary hyperbolic 3-manifold M, the index of a subgroup of $\pi_1 M$ that embeds in a right-angled Artin group. However we note that the groups Γ of Theorem 2.3 all embed $\mathrm{SO}(6,1;\mathbb{Z})$, which is itself virtually special (by [10] or [4]). We believe it would be interesting to find a subgroup of $\mathrm{SO}(6,1;\mathbb{Z})$ embedded in a right-angled Artin group and, using this, to produce explicit residual finiteness growth bounds for $\pi_1 M$ along the lines of Corollary 3.3.

1. Excluding group elements with right-angled polyhedra

Lemma 1.1. *For a right-angled polyhedron* $P \subset \mathbb{H}^{n+1}$*, an ideal vertex* v *of* P*, and a horoball* B *centered at* v *and embedded in* P *(in the sense of Definition 1), if* Γ_P *is the group of generated by reflections in the sides of* P *then for* $\gamma \in \Gamma_P$*,* $B \cap \gamma.B \neq \emptyset$ *if and only if* γ *lies in the stabilizer* $\Gamma_P(v)$ *of* v *in* Γ_P*.*

Proof. Since B is embedded in P, $P \cap \partial B$ is a right-angled polyhedron in ∂B, which inherits a Riemannian metric isometric to the Euclidean metric on $\mathbb{R}^n$ from $\mathbb{H}^{n+1}$. Therefore by the Euclidean case of the Poincaré polyhedron theorem (see eg. [19, Theorem 13.5.3]), ∂B is tiled by translates of $P \cap \partial B$ under the action of the group generated by reflections in its sides. Each such reflection is the restriction to ∂B of the reflection of $\mathbb{H}^n$ in a side of P that contains v; in particular, in an element of $\Gamma_P(v)$. It follows that:

$$B \subset \bigcup \{\, \gamma.P : \gamma \in \Gamma_P(v) \,\}. \tag{1}$$

Now suppose for some $\gamma \in \Gamma_P$ that $B \cap \gamma.B \neq \emptyset$, and let x be a point in the intersection. Let $v' = \gamma.v$ be the ideal point of $\gamma.B$. Applying the above to B and $\gamma.B$ yields $\lambda_0 \in \Gamma_P(v)$ and $\lambda_1 \in \Gamma_P(v') = \gamma\Gamma_P(v)\gamma^{-1}$ such that $\lambda_0^{-1}.x$ and $\gamma^{-1}\lambda_1^{-1}.x$ lie in $B \cap P$. Thus $\gamma^{-1}\lambda_1^{-1}\lambda_0$ takes $B \cap P$ to intersect itself. Since P is a fundamental domain for Γ_P and B is embedded in P this implies that $\gamma^{-1}\lambda_1^{-1}\lambda_0^{-1}$ is either the identity or the reflection in a side of P containing v. In any case it follows that $\gamma \in \Gamma_P(v)$, since $\lambda_1 = \gamma\lambda_1'\gamma^{-1}$ for some $\lambda_1' \in \Gamma_P(v)$. □

For M as in Theorem 1.6, the totally geodesic immersion $f\colon M \to \mathcal{O}_P$ lifts to a totally geodesic embedding from the universal cover of M to an m-dimensional hyperplane of $\mathbb{H}^{n+1}$, to which we will refer by $\mathbb{H}^m$. This map is equivariant with respect to the actions of $\pi_1 M$ and $f_*(\pi_1 M) \subset \Gamma_P$ by covering transformations, so we will regard $\pi_1 M$ as a subgroup of Γ_P that stabilizes $\mathbb{H}^m$ and acts cocompactly on it. The map from $\mathbb{H}^m$ to $\mathcal{O}_P$ factors as f composed with the quotient map $\mathbb{H}^m \to \mathbb{H}^m/\pi_1 M$, which we will call the *universal cover*.

Lemma 1.2. *With the hypotheses of Theorem 1.6, for each $B \in \mathcal{B}$ and $\gamma \in \Gamma_P$, if $\gamma.B \cap \mathbb{H}^m$ is non-empty then it is a compact metric ball in $\mathbb{H}^m$ with radius r_h satisfying $\cosh r_h = e^{h_{\gamma.B}}$, where $h_{\gamma.B}$ is the maximum, taken over all $x \in \gamma.B \cap \mathbb{H}^m$, of the distance from x to ∂B. The interior of this ball embeds in M under the universal cover.*

Proof. The boundary at infinity of $\mathbb{H}^m$ does not contain an ideal point of any Γ_P-translate of P: if it did then $\pi_1 M$, which acts preserving the tiling of $\mathbb{H}^m$ by its intersection with such translates, would have a non-compact fundamental domain, contradicting cocompactness. Since the horoballs $\gamma.B \cap \mathbb{H}^m$ are each centered at such points, for each such γ, $\mathbb{H}^m$ does not contain the ideal point of $\gamma.B$.

Suppose now that $\mathbb{H}^m$ does intersect $\gamma.B$ for some $\gamma \in \Gamma_P$. Lemma 1.1 implies that for any $\lambda \in \pi_1 M$ that takes $\gamma.B$ to overlap itself, λ lies in the stabilizer $\Gamma_P(\gamma.v)$ of the ideal point $\gamma.v$ of $\gamma.B$. But all such elements are parabolic, and $\pi_1 M$ has no parabolic elements since it acts cocompactly. It follows that the interior of $\gamma.B \cap \mathbb{H}^m$ embeds in M under the universal cover.

Working in the Poincaré ball model $\mathbf{D}^{n+1}$ for $\mathbb{H}^{n+1}$, we translate $\mathbb{H}^m$ and $\gamma.B$ by isometries so that $\mathbb{H}^m = \mathbf{D}^m \times \{\mathbf{0}\}$ and the ideal point of $\gamma.B$ is at $(0, \ldots, 0, 1)$. Then $\gamma.B$ is a Euclidean ball with radius $r \in [1/2, 1)$ and Euclidean center $(0, \ldots, 0, 1-r)$. By the Pythagorean theorem, $\gamma.B$ therefore intersects $\mathbf{D}^m \times \{\mathbf{0}\}$ in a Euclidean ball of radius $\sqrt{2r-1}$ in $\mathbf{D}^m$, centered at $\mathbf{0}$. We now recall the formula for the hyperbolic distance d in $\mathbf{D}^n$ (see eg. [19, Theorem 4.5.1]:

$$\cosh d(\mathbf{x}, \mathbf{y}) = 1 + \frac{2|\mathbf{x} - \mathbf{y}|^2}{(1 - |\mathbf{x}|^2)(1 - |\mathbf{y}|^2)},$$

where $|\cdot|$ is the Euclidean norm. Therefore the hyperbolic radius r_h of the ball of intersection satisfies $\cosh r_h = r/(1-r)$. On the other hand, some manipulation shows that the hyperbolic distance h from $\mathbf{0}$ to the lowest point $(0, \ldots, 0, 1-2r)$ of B satisfies $e^h = r/(1-r) = \cosh r_h$. And this is the closest point of ∂B to $\mathbf{0}$, since the formula above gives $\cosh d(\mathbf{0}, \mathbf{y}) = 1 + 2|\mathbf{y}|^2/(1 - |\mathbf{y}|^2)$ for any $\mathbf{y} \in \partial B$. This increases with $|\mathbf{y}|^2$, which in turn increases with y_n, as can be discerned by rearranging the equation $|\mathbf{y} - (0, \ldots, 0, 1-r)|^2 = r^2$ to:

$$|\mathbf{y}|^2 = 2r - 1 + 2(1-r)y_n.$$

Given any $\mathbf{x} = (\mathbf{x}_0, 0) \in \mathbb{H}^m \times \{\mathbf{0}\}$, there is a unique point $\mathbf{y} = (\mathbf{x}_0, y) \in \partial B$ "directly below $\mathbf{x}$", that is, with $y < 0$. A direct computation now shows that the distance from $\mathbf{x}$ to $\mathbf{y}$ decreases with $|\mathbf{x}|^2$, so $\mathbf{0}$ is the furthest point of $\mathbb{H}^m \cap B$ from ∂B and the lemma is proved. □

For a horoball B of $\mathbb{H}^n$ and a totally geodesic hyperplane $\mathbb{H}^m \subset \mathbb{H}^n$ that is not incident on the ideal point v of B, define the *height* of $\mathbb{H}^m$ with respect to B to be the maximal signed distance from points of $\mathbb{H}^m$ to ∂B, where the sign is non-negative for points of $\mathbb{H}^m \cap B$.

Corollary 1.3. *For P, $\mathcal{B}$, and M as in Theorem 1.6, if $f_*(\pi_1 M)$ stabilizes $\mathbb{H}^m$ then for any $B \in \mathcal{B}$ and $\gamma \in \Gamma_P$, the height $h_{\gamma.B}$ of $\mathbb{H}^m$ with respect to $\gamma.B$ satisfies $e^{h_{\gamma.B}} \leq \cosh r_{\max}$, where $r_{\max}$ is the maximal radius of a ball embedded in M.*

Below, for a fixed right-angled polyhedron $P \subset \mathbb{H}^{n+1}$ we will call the *convexification* of a set $\mathcal{K} \subset \mathbb{H}^{n+1}$ the P-convexification from [16, Definition 2.1]: it is the minimal convex union of Γ_P-translates of P containing $\mathcal{K}$.

Lemma 1.4. *For $P \subset \mathbb{H}^{n+1}$, M, and α as in Theorem 1.6, let $\tilde{\alpha}$ be the geodesic axis in $\mathbb{H}^{n+1}$ of $f_*(\alpha)$. Any polyhedron P_i in the convexification of $\tilde{\alpha}$ intersects the R-neighborhood of $\tilde{\alpha}$, where $R = \ln(\sqrt{n+1} + \sqrt{n})$.*

Proof. The proof follows the strategy of Lemmas 3.1 and 4.2 of [16], which respectively establish the cases $n = 2$ and $n = 3$ (i.e. where P is 3- or 4-dimensional). As in those proofs we work in the ball model $\mathbf{D}^{n+1}$ for $\mathbb{H}^{n+1}$ and fix a Γ_P-translate of P (which we will again just call P) that does not intersect the R-neighborhood of $\tilde{\alpha}$. The goal is to show that $\tilde{\alpha}$ and P are on opposite sides of a hyperplane containing one of the faces of P, from which it follows that P is not in the convexification.

We suppose first that the closest point of P to $\tilde{\alpha}$ is a vertex e, and move the entire picture by isometries so that e lies at the origin. The sides of P that contain e are contained in totally geodesic hyperplanes, each of which is the intersection of a Euclidean hyperplane with $\mathbf{D}^{n+1}$ since it contains the origin. Their intersections with $\mathbb{S}^n$ divide it into right-angled spherical simplices. The key computation here is the in-radius of such a simplex; that is, the minimum radius of a metric sphere in $\mathbb{S}^n$ that intersects every hyperplane.

Claim. An all-right simplex in $\mathbb{S}^n$ has in-radius $\theta = \cos^{-1}\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right)$.

Deferring the claim's proof for the moment, we describe its application to our situation following [16, Lemma 3.1]. Let j be the geodesic hyperplane containing $\tilde{\alpha}$ that is perpendicular to the arc $\overline{0y}$ from e (which we have moved to 0) to the closest point y to e on α. The fact that $d(e, \tilde{\alpha}) > R$ for $R = \ln(\sqrt{n+1} - \sqrt{n})$ ensures that j intersects $\partial\mathbf{D}^{n+1} = \mathbb{S}^n$ in a sphere of radius (in the spherical metric) less than $\cos^{-1}\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right)$, by a calculation entirely analogous to the one spanning pp. 93–94 of [16]. In particular, the "cross sectional view" of Figure 3 there still holds (the cross section just has higher codimension). This sphere is therefore disjoint from the intersection with $\mathbb{S}^n$ of at least one hyperplane containing a side of P that contains e. It follows as in [16] that this hyperplane separates $\tilde{\alpha}$ from P.

Proof of claim. After applying a sequence of orthogonal transformations we may take the given hyperplanes to be the intersections with $\mathbb{S}^n$ of the coordinate planes in $\mathbb{R}^{n+1}$: apply an orthogonal transformation that moves the first hyperplane's normal vector to $\mathbf{e}_1$, then apply an orthogonal transformation of $\mathbf{e}_1^\perp$ that moves the second hyperplane's normal vector to $\mathbf{e}_2^\perp$, etc. The coordinate hyperplanes divide $\mathbb{S}^n$ into right-angled simplices, each with the property that for any two of its points, the i^{th} entry of the first has the same sign as the i^{th} entry of the second for each $i \in \{1, \ldots, n+1\}$. We restrict our attention to the simplex σ_n consisting of points with all entries non-negative, noting that any of the others is isometric to σ_n by a map which simply multiplies each entry by ± 1.

Note that the symmetric group S_{n+1} acts isometrically on $\mathbb{S}^n$ by permuting entries, preserving σ_n and acting transitively on its set of faces of dimension k, for any fixed $k < n$. The barycenter of σ_n, the sole global fixed point in σ_n of this action, is $\mathbf{v}_n = \frac{1}{\sqrt{n+1}}(1, \ldots, 1)$. Similarly call $\mathbf{v}_k$ the barycenter of $\sigma_k \subset \mathbb{S}^k$ for each $k < n$. Upon including σ_k in σ_n by the map $\mathbb{R}^{k+1} \to \mathbb{R}^{k+1} \times \{\mathbf{0}\} \subset \mathbb{R}^n$, we directly compute the spherical distance $d(\mathbf{v}_n, \mathbf{v}_k)$ from $\mathbf{v}_n$ to $\mathbf{v}_k$ via:

$$\cos d(\mathbf{v}_n, \mathbf{v}_k) = \left[\frac{1}{\sqrt{n+1}}(1, \ldots, 1)\right] \cdot \left[\frac{1}{\sqrt{k+1}}(\overbrace{1, \ldots, 1}^{k+1}, 0, \ldots, 0)\right] = \frac{\sqrt{k+1}}{\sqrt{n+1}}$$

A straightforward direct argument now shows that $\mathbf{v}_k$ is the closest point of σ_k to σ_n. For each $\mathbf{x} = (x_1, \ldots, x_{k+1}, 0, \ldots, 0) \in \sigma_k$, $\mathbf{x} \cdot \mathbf{v}_n = \mathbf{x} \cdot \pi(\mathbf{v}_n)$, where $\pi(\mathbf{v}_n) = \frac{1}{\sqrt{n+1}}(1, \ldots, 1, 0, \ldots, 0)$ is the projection of $\mathbf{v}_n$ to $\mathbb{R}^{k+1} \times \{\mathbf{0}\}$. The Cauchy-Schwarz inequality asserts that $\mathbf{x} \cdot \pi(\mathbf{v}_n) \leq \|\mathbf{x}\|\|\pi(\mathbf{v}_n)\| = \frac{\sqrt{k+1}}{\sqrt{n+1}}$, with equality holding if and only if $\mathbf{x}$ is a scalar multiple of $\pi(\mathbf{v}_n)$; that is, $\mathbf{v}_k$. Since the inverse cosine is a decreasing function, the assertion follows.

We note in particular that $d(\mathbf{v}_n, \mathbf{v}_k)$ decreases with k. So the closest points to $\mathbf{v}_n$ on $\partial\sigma_n$, which is a union of S_{n+1}-translates of σ_{n-1}, are the S_{n+1}-translates of $\mathbf{v}_{n-1}$. Therefore the metric sphere of radius $\cos^{-1}\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right)$ centered at $\mathbf{v}_n$ is inscribed in σ_n and tangent to $\partial\sigma_n$ at each S_{n+1}-translate of σ_{n-1}. In particular, this sphere intersects every side of σ_n.

To establish the claim it remains to show for each $\mathbf{v} \in \sigma_n$ that there is some side of σ_n that is at least as far from $\mathbf{v}$ as from $\mathbf{v}_n$. To this point we note that if $\mathbf{v} = (v_1, \ldots, v_{n+1}) \in \sigma_n - \{\mathbf{e}_{n+1}\}$ then the closest point of σ_{n-1} to $\mathbf{v}$ is $\mathbf{x} = \pi(\mathbf{v})/\|\pi(\mathbf{v})\|$, where $\pi(\mathbf{v}) = (v_1, \ldots, v_n)$. This follows from the Cauchy-Schwarz inequality as above. We compute that $\pi(\mathbf{v}) \cdot \pi(\mathbf{v}) = v_1^2 + \ldots + v_n^2 = 1 - v_{n+1}^2$, so

$$d(\mathbf{v}, \sigma_{n-1}) = d(\mathbf{v}, \mathbf{x}) = \cos^{-1}\left(\frac{\mathbf{v} \cdot \pi(\mathbf{v})}{\|\pi(\mathbf{v})\|}\right) = \cos^{-1}\sqrt{1 - v_{n+1}^2}.$$

(This formula also holds for $\mathbf{v} = \mathbf{e}_{n+1}$, which has distance $\pi/2 = \cos^{-1}(0)$ from all points of σ_{n-1}.) Each other side of σ_n is also contained in a coordinate plane; call $\sigma_{n-1}^{(i)}$ the side contained in the coordinate plane perpendicular to $\mathbf{e}_i$ (so $\sigma_{n-1} = \sigma_{n-1}^{(n+1)}$). For each $\mathbf{v} \in \sigma$ and $1 \leq i \leq n+1$, an analogous argument shows that

$$d(\mathbf{v}, \sigma_{n-1}^{(i)}) = \cos^{-1}\sqrt{1 - v_i^2}.$$

The right side of this equation increases with v_i, so for fixed $\mathbf{v}$ the distance to $\sigma_{n-1}^{(i)}$ is maximized at any i for which v_i is maximal. But the maximum entry of $\mathbf{v}$ is at least $1/\sqrt{n+1}$ since $\|\mathbf{v}\| = 1$. $\square$

It remains to consider the case when the nearest point of P to $\tilde{\alpha}$ is not a vertex. We handle this case by induction, more or less: if the closest point p of P to $\tilde{\alpha}$ lies in the interior of a face e of codimension $k \leq n$ then we work in the k-dimensional geodesic subspace L of $\mathbb{H}^{n+1}$ that contains p and is orthogonal to the $(n+1-k)$-plane containing e. For each side of P that contains e, the hyperplane containing it intersects L perpendicularly in a codimension-one geodesic subspace, and the collection of all these subspaces determines a polyhedron in L which contains $P \cap L$ and has a single vertex at p. This polyhedron intersects ∂L in an all-right spherical simplex of dimension $k-1$, which by the claim has in-radius $\cos^{-1}\left(\frac{\sqrt{k-1}}{\sqrt{k}}\right)$.

This quantity is larger than $\cos^{-1}\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right)$, so for j as above it follows that the intersection with L of at least one hyperplane containing a side of P does not intersect $j \cap L$. Since both j and this hyperplane intersect L orthogonally, it follows that j misses this hyperplane, which hence again separates $\tilde{\alpha}$ from P. □

Lemma 1.5. *A tubular neighborhood in $\mathbb{H}^{n+1}$ of radius R around a geodesic segment of length ℓ has volume $\mathrm{Vol}(\mathbf{B}^n)\sinh^n(R)\ell$, where $\mathrm{Vol}(\mathbf{B}^n)$ is the Euclidean volume of the unit ball in $\mathbb{R}^n$.*

Proof. This is a straightforward generalization of Lemmas 3.2 and 4.1 of [16]. Details are worked out in the preprint version [15] of [16], see Lemma 6.2 there. □

Theorem 1.6. *For $n \geq 2$, let P be a right-angled polyhedron in $\mathbb{H}^{n+1}$ with finite volume and at least one ideal vertex, let Γ_P be the group generated by reflections in the sides of P, and let $\mathcal{B}$ be a collection of horoballs, one for each ideal vertex of P, that are each embedded in the sense of Definition 1 and pairwise non-overlapping. For a closed hyperbolic m-manifold M, $m \leq n$, that admits a totally geodesic immersion to $\mathcal{O}_P \doteq \mathbb{H}^{n+1}/\Gamma_P$, and any $\alpha \in \pi_1 M - \{1\}$, there exists a subgroup H' of $\pi_1 M$ such that $\alpha \notin H'$, and the index of H' is bounded above by*

$$\frac{2v_n(1)}{V_{R+h_{\max}}}\sinh^n\left(R + d_{R+h_{\max}}\right)\ell(\alpha),$$

where $v_n(1)$ is the (Euclidean) volume of the n-dimensional Euclidean unit ball and:

- *$\ell(\alpha)$ is the length of the unique geodesic representative of α;*
- *$R = \ln(\sqrt{n+1} + \sqrt{n})$;*
- *$h_{\max} = \ln(\cosh r_{\max})$, where $r_{\max}$ is the radius of the largest embedded ball in M; and*
- *$d_{R+h_{\max}}$ and $V_{R+h_{\max}}$ are the diameter and volume, respectively, of the $(R + h_{\max})$-neighborhood in P of $\overline{P - \bigcup\{B \in \mathcal{B}\}}$.*

Proof. With hypotheses as in Theorem 1.6, let $\tilde{\alpha} \subset \mathbb{H}^m$ be the geodesic axis of α, where $\mathbb{H}^m$ is the totally geodesic subspace of $\mathbb{H}^{n+1}$ stabilized by $\pi_1 M$. We claim that every polyhedron $\gamma.P$ in the convexification $\mathcal{C}$ of $\tilde{\alpha}$ has its closest point to $\tilde{\alpha}$ in $\gamma.\mathcal{N}_{R+h_{\max}}$, where $\mathcal{N}_{R+h_{\max}}$ is the $(R + h_{\max})$-neighborhood of $\overline{P - \bigcup\{B \in \mathcal{B}\}}$.

Suppose, for some $\gamma \in \Gamma_P$ such that $\gamma.P$ is in the convexification of $\tilde{\alpha}$, that the nearest point x of $\gamma.P$ to $\tilde{\alpha}$ lies in $\gamma.B$, for some $B \in \mathcal{B}$, at distance greater than R from $\partial(\gamma.B)$. Then the nearest point y on $\tilde{\alpha}$ to x also lies in $\gamma.B$, by Lemma 1.4. But by Corollary 1.3, y is no further from $\partial(\gamma.B)$ than $h_{\max}$, so x lies no further than $R + h_{\max}$ from $\partial(\gamma.B)$. This proves the claim.

The claim now implies for each translate $\gamma.P$ in $\mathcal{C}$ that all of $\gamma.\mathcal{N}_{R+h_{\max}}$ is contained in the $(R + d_{R+h_{\max}})$-neighborhood of $\tilde{\alpha}$. We now obtain the bound of the theorem by arguing as in the proof of [16, Theorem 3.3]. □

2. Arithmetic lattice bounds

Given a $\mathbb{Q}$–defined bilinear form q of signature $(3,1)$, any group $\Gamma < \mathrm{SO}(3,1)$ that is commensurable with $\mathrm{SO}(q,\mathbb{Z})$ provides an arithmetic lattice in the isometry group of hyperbolic 3–space. When q is anistropic over $\mathbb{Q}$ (i.e. $q(v,v) \neq 0$ for every $v \in \mathbb{Q}^4$), the associated lattice is cocompact by work of Borel–Harish-Chandra [6, §11]. When q is isotropic, it follows that $\mathrm{SO}(q,\mathbb{Z})$ is commensurable with a Bianchi group $\mathrm{PSL}(2,\mathcal{O}_K)$ where $K/\mathbb{Q}$ is an imaginary quadratic extension (see

[12]). However, there exist infinitely many commensurability classes of lattices coming from such forms above that are cocompact.

Given an arithmetic lattice Γ in the commensurability class of $\mathrm{SO}(q,\mathbb{Z})$, it follows that $[\Gamma : \Gamma \cap \mathrm{SO}(q,\mathbb{Z})] = C$ for some C. To determine an upper bound for C, we require work of Borel–Prasad [7] and Prasad [18]. First, by Borel–Prasad [7, Prop. 1.4 (iii)], we know that Γ is contained in only finitely many maximal arithmetic lattices in the commensurability class containing $\mathrm{SO}(q,\mathbb{Z})$; any maximal lattices containing Γ is visibly commensurable with $\mathrm{SO}(q,\mathbb{Z})$. Borel–Prasad [7, Prop. 1.4 (iv)] classified of the maximal arithmetic lattices and these lattices arise as normalizers of principal arithmetic subgroups of $\mathrm{SO}(q,\mathbb{Z})$. The principal arithmetic lattices in $\mathrm{SO}(q,\mathbb{Z})$ are in bijection with coherent systems of parahoric subgroups $(\mathrm{P}_v)_{v\in V_{\mathbb{Q}}^f}$ where v varies over all of the non-archimedean places $V_{\mathbb{Q}}^f$ of $\mathbb{Q}$. Setting p_v to be the finite prime associated $v \in V_{\mathbb{Q}}^f$, each parahoric subgroup P_v is compact open subgroups of $\mathrm{SO}(q,\mathbb{Z}_{p_v})$ where $\mathbb{Z}_{p_v}$ is the closure of $\mathbb{Z}$ inside the complete local field $\mathbb{Q}_{p_v}$; this is just the field of p_v–adic numbers. The principal arithmetic lattice Γ_{P} associated to the coherent system of parahoric subgroups $\mathrm{P} = (\mathrm{P}_v)_{v\in V_{\mathbb{Q}}^f}$ is defined by

$$\Gamma_{\mathrm{P}} = \mathrm{SO}(q,\mathbb{Q}) \cap \prod_{v\in V_{\mathbb{Q}}^f} \mathrm{P}_v$$

where $\mathrm{SO}(q,\mathbb{Q})$ is embedded into $\mathrm{SO}(q,\mathbb{A}_f)$ diagonally and $\mathbb{A}_f = \prod_{v\in V_{\mathbb{Q}}^f} \mathbb{Q}_{p_v}$ is the ring of finite $\mathbb{Q}$–adeles. There are only finitely many places $v \in V_{\mathbb{Q}}^f$ where $\mathrm{P}_v \neq \mathrm{SO}(q,\mathbb{Z}_{p_v})$ and we set $v_1,\dots,v_r$ be the set of such places. It follows from the volume formula of Prasad [18, Thm 3.7] (see also [3, §2] or [5] for a discussion on this particular setting) that the co-volume of the principal arithmetic lattice Γ_{P} associated to the coherent system of parahoric subgroups $\mathrm{P} = (\mathrm{P}_v)_{v\in V_{\mathbb{Q}}^f}$ satisfies

$$\mathrm{covol}(\Gamma_{\mathrm{P}}) \geq \left(\prod_{i=1}^r p_{v_i} - 1\right) \cdot \mathrm{covol}(\mathrm{SO}(q,\mathbb{Z})).$$

In particular, since $p_{v_i} - 1 \geq 3$ except when $p_{v_i} = 2$ or 3, we see that

$$\mathrm{covol}(\Gamma_{\mathrm{P}}) \geq \left(3^{r-2}\right) \mathrm{covol}(\mathrm{SO}(q,\mathbb{Z})).$$

Taking Λ_{P} to be the normalizer of Γ_{P} in $\mathrm{SO}(3,1)$, Borel–Prasad [7, §2] (see also [3, §4] for a discussion on the index of these normalizers in this particular setting) determined that $[\Lambda_{\mathrm{P}} : \Gamma_{\mathrm{P}}] \leq 2^r$. Briefly, at each place $v \neq v_1,\dots,v_r$, since $\mathrm{P}_v = \mathrm{SO}(q,\mathbb{Z}_{p_v})$ and the latter is self-normalizing, the index $[\Lambda_{\mathrm{P}} : \Gamma_{\mathrm{P}}]$ can only increase at the places $v_1,\dots,v_r$. At each place v_i, we can increase this index by at most 2 since $\mathrm{SO}(3,1)$ is type D_2. Returning to our current task, as $\Gamma_{\mathrm{P}} \leq \mathrm{SO}(q,\mathbb{Z})$, we see that if $\Gamma \leq \Lambda_{\mathrm{P}}$, then

$$[\Gamma : \Gamma \cap \mathrm{SO}(q,\mathbb{Z})] \leq 2^r.$$

Moreover, we see that

$$\mathrm{covol}(\Lambda_{\mathrm{P}}) \geq \left(\frac{3^{r-2}}{2^r}\right) \mathrm{covol}(\mathrm{SO}(q,\mathbb{Z})).$$

Since $\mathrm{covol}(\Gamma) \geq \mathrm{covol}(\Lambda_{\mathrm{P}})$, we deduce that

$$\mathrm{covol}(\Gamma) \geq \left(\frac{3^{r-2}}{2^r}\right) \mathrm{covol}(\mathrm{SO}(q,Z)). \tag{2}$$

Let r be the largest integer such (2) holds. Solving for r, we see that

$$\begin{aligned}\frac{\mathrm{covol}(\Gamma)}{\mathrm{covol}(\mathrm{SO}(q,\mathbb{Z}))} &\geq \frac{3^{r-2}}{2^r}\\ \log(\mathrm{covol}(\Gamma)) - \log(\mathrm{covol}(\mathrm{SO}(q,\mathbb{Z})) &\geq r(\log(3)-\log(2)) - 2\log(3)\\ \frac{\log(\mathrm{covol}(\Gamma)) - \log(\mathrm{covol}(\mathrm{SO}(q,\mathbb{Z})) + 2\log(3)}{\log(3)-\log(2)} &\geq r\end{aligned}$$

Taking C to be the minimal volume of an arithmetic hyperbolic 3–orbifold, we obtain

$$r \leq \frac{\log(\mathrm{covol}(\Gamma)) - C + 2\log(3)}{\log(3)-\log(2)}.$$

Hence

$$[\Gamma : \Gamma \cap \mathrm{SO}(q,\mathbb{Z})] \leq 2^r \leq \left(\frac{2^{(2\log(3)-C)/(\log(3)-\log(2))}}{\log(3)-\log(2)}\right)\mathrm{covol}(\Gamma).$$

We summarize this upper bound in the following lemma.

Lemma 2.1. *Let Γ be an arithmetic lattice that is commensurable with $\mathrm{SO}(q,\mathbb{Z})$ for some $\mathbb{Q}$–defined bilinear form q of signature $(3,1)$. Then there exists an absolute constant C_1 such that*

$$[\Gamma : \Gamma \cap \mathrm{SO}(q,\mathbb{Z})] \leq C_1 \mathrm{covol}(\Gamma).$$

The constant C_1 can be taken to be

$$\frac{2^{(2\log(3)-C)/(\log(3)-\log(2))}}{\log(3)-\log(2)}$$

where C is the minimal volume of an arithmetic hyperbolic 3–orbifold.

We next show the following lemma.

Lemma 2.2. *There exists a constant D such that there exists a subgroup $\Delta \leq \mathrm{SO}(q,\mathbb{Z})$ of index at most D and an injective homomorphism $\Delta \to \mathrm{SO}(6,1;\mathbb{Z})$. Moreover, $D \leq x^{21}$ where x is explicitly determined by q.*

Before proving Lemma 2.2, we deduce the main result of this section.

Theorem 2.3. *Let Γ be an arithmetic lattice that is commensurable with $\mathrm{SO}(q,\mathbb{Z})$ for some $\mathbb{Q}$–defined bilinear form q. Then there exists an absolute constant C_1 and a constant D that depends only on the commensurability class of Γ such that Γ has a subgroup Δ of index at most $C_1 D \mathrm{covol}(\Gamma)$ and an injective homomorphism from Δ to a subgroup of $\mathrm{SO}(6,1;\mathbb{Z})$ that stabilizes a time-like subspace of $\mathbb{R}^{6,1}$.*

Thus for an arithmetic hyperbolic manifold $M = \mathbb{H}^3/\Gamma$, where Γ is such a lattice, there is a cover $\tilde{M} \to M$ of degree at most $C_1 D \mathrm{vol}(M)$ with a totally geodesic immersion to $\mathbb{H}^6/\Gamma_P$, for a right-angled polyhedron P.

Theorem 2.3 follows immediately from Lemma 2.1 and Lemma 2.2. For completeness, we provide the details of this deduction.

Proof of Theorem 2.3. By Lemma 2.1, we know that $[\Gamma : \Gamma \cap \mathrm{SO}(q,\mathbb{Z})] \leq C_1 \mathrm{covol}(\Gamma)$ for an absolute constant C_1. By Lemma 2.2, there exists a subgroup $\Delta \leq \mathrm{SO}(q,\mathbb{Z})$ of index at most D, where D is a constant that depends only on q, such that Δ admits an injective homomorphisms into $\mathrm{SO}(6,1;\mathbb{Z})$. As

$$[\Gamma \cap \mathrm{SO}(q,\mathbb{Z}) : \Gamma \cap \mathrm{SO}(q,\mathbb{Z}) \cap \Delta] \leq D,$$

we conclude that there is a subgroup of index at most $C_1 D\mathrm{covol}(\Gamma)$ of Γ that admits an injective homomorphism into $\mathrm{SO}(6,1;\mathbb{Z})$.

We note that the injective homomorphism of the finite index subgroup of Γ into $\mathrm{SO}(6,1;\mathbb{Z})$ induces a totally geodesic immersion of the associated arithmetic hyperbolic orbifolds. As noted in the introduction, the Theorem's second paragraph now follows from [1, Lemma 3.4]. □

It remains to prove Lemma 2.2. We first prove the existence of D and then will provide the stated upper bound as a function of q.

Proof of Lemma 2.2. To start, we can assume that $q = \langle z_1, z_2, z_3, z_4\rangle$, where each z_i is square-free integer. We fix the form $q_0 := \langle 1,1,1,1,1,1,-1\rangle$ on $\mathbb{Q}^7$ and the lattice $L := \mathbb{Z}^7 \subset \mathbb{Q}^7$. By [14] (see also [13, Prop. 7.16]), there exists a lattice $L_q \subseteq L$ such that the restriction of q_0 to L_q is $\mathbb{Q}$-isometric to $\langle 1,1,1\rangle \oplus q$. From this, the existence of D in the lemma follows immediately. □

In order to give an estimate for D in Lemma 2.2, we need to bound the index $[L : L_q]$. To this end, we briefly review the classification of bilinear forms over $\mathbb{Q}$. As our forms are all non-degenerate and all have determined signatures, the two main invariants for the classification of forms are the $\det(q)$ which is an element of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ and the Hasse invariant A_q which is an element of the Brauer group $\mathrm{Br}(\mathbb{Q})$ of $\mathbb{Q}$. For each prime p, the local invariant $\mathrm{Inv}_p(A_q)$ for A_q is ± 1 and we say A_q is ramified at p when $\mathrm{Inv}_p(A_q) = -1$. By The Albert–Brauer–Hasse–Noether theorem (see [12]), A_q is ramified at finitely many primes and is completely determined by the set $\mathrm{Ram}(A_q)$ of such primes.

Returning to our estimate of D, by [14] (see also [13, §7]), there exists a definite quadratic form r of signature $(3,0)$ such that

$$r \oplus \langle 1,1,1,-1\rangle \cong \langle 1,1,1\rangle \oplus q.$$

This form is uniquely determined by the conditions

$$\det r = -d := -\det q \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \tag{3}$$

$$\mathrm{Inv}_p(r) = \mathrm{Inv}_p(A_q)\mathrm{Inv}_p(A_{-d,-1}) \tag{4}$$

where $A_{-d,-1}$ is the quaternion algebra with Hilbert symbol $\left(\frac{-d,-1}{\mathbb{Q}}\right)$. Additionally, we can assume that r is diagonal and of the form $\langle a,b,c\rangle$, where $a,b,c \in \mathbb{N}$ are square-free. It follows that r is the restriction of the form $\langle 1,1,1\rangle$ to the index abc sublattice $\Lambda_q := a\mathbb{Z} \oplus b\mathbb{Z} \oplus c\mathbb{Z}$ of the lattice $\Lambda := \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. Let

$$S = \{p \in \mathbb{N} \text{ prime } : \text{ either } p \mid \det q \text{ or } p \in \mathrm{Ram}(A_q)\} \cup \{2\}$$

and let

$$x = \prod_{p\in S} p = 2\left(\prod_{p\mid \det q} p\right)\left(\prod_{\substack{p\in\mathrm{Ram}(A_q)\\ p\nmid \det q}} p\right). \tag{5}$$

Equation (4) implies that each $a,b,c|x$. Finding a,b,c as a function of $\det q$ and $\mathrm{Inv}_p(A_q)$ is rather difficult, however it follows that since each entry divides x, we need only go to as deep as the index x^3 sublattice $\Lambda_x := x\mathbb{Z}\oplus x\mathbb{Z}\oplus x\mathbb{Z}$ to guarantee $\Lambda_x \subseteq \Lambda_q \subseteq \Lambda$ to find a lattice that $\langle 1,1,1,1,1,1,-1\rangle$ restricted to that lattice is isometric to $\langle 1,1,1\rangle \oplus q$. Direct summing with $\langle 1,1,1,-1\rangle$, we obtain $L_x \subseteq L_q \subseteq L$

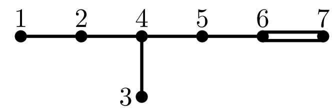


FIGURE 1. The Coxeter diagram of a simplex $\sigma \subset \mathbb{H}^6$.

and hence $[L : L_q]$ divides x^3. From the construction of the forms above, it follows that the stabilizer of L_q in $\mathrm{SO}(6,1;\mathbb{Z}) = \mathrm{SO}(q_0,\mathbb{Z})$ has index bounded above by x^{21}; here $\dim \mathrm{SO}(6,1) = 21$. In total, this yields a subgroup of $\mathrm{SO}(q,\mathbb{Z})$ of index at most x^{21} that embeds into $\mathrm{SO}(q_0;\mathbb{Z})$.

Examples. For the readers' clarity, we give a few explicit examples of q and determine the value of x in each case.

- If $q = \langle 1,1,1,-1\rangle$, we see that $\det(q) = -1$ and that the Hasse invariant is trivial. In this case, we can take $x = 1$. Indeed, q is visibly a factor of $\langle 1,1,1,1,1,1,-1\rangle$.
- If $q = \langle 1,1,1,-p\rangle$, we see that $\det(q) = -p$ and that the Hasse invariant is trivial. In this case, we can take $x = p$.

3. Explicit constants

We begin by extracting relevant geometric data on the simplex $\sigma \subset \mathbb{H}^6$ discussed in the introduction, which is a fundamental domain for the action of $\mathrm{SO}(6,1;\mathbb{Z})$. The Coxeter diagram of σ, reproduced in Figure 1 with vertices numbered, encodes the angles of intersections between the sides of σ. It has a vertex for each side of σ, with two vertices connected by a single edge if their corresponding sides intersect with an interior angle of $\pi/3$. The sides corresponding to the two vertices connected by the doubled edge intersect with an interior angle of $\pi/4$. Two vertices are not joined by an edge if the sides they represent intersect at right angles.

Theorem 7.3.1 of [19] gives a constructive proof of the existence of Coxeter simplices, building on Theorem 7.2.4 there. We will follow these proofs to give an explicit description of σ in the hyperboloid model for $\mathbb{H}^6$ (see eg. [19, Ch. 3] for an introduction to this model). For each i between 1 and 7, let S_i be the side of σ corresponding to the vertex labeled i. We will first locate the inward-pointing normal $\mathbf{v}_i$ to S_i for each such i. Then for each i we will locate the vertex $\mathbf{x}_i$ of σ opposite S_i. (We are following Ratcliffe's notation as closely as possible here; note in particular that $\mathbf{v}_i$ is *not* a vertex of σ.)

The Gram matrix A of σ can be read off from the Coxeter diagram. Its (i,j)-entry is $-\cos\theta_{ij}$, where θ_{ij} is the interior angle of σ at $S_i \cap S_j$.

$$A = \begin{pmatrix} 1 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & -1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 1 & -1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -1/\sqrt{2} & 1 \end{pmatrix}$$

Applying the Gram–Schmidt process to the standard basis of $\mathbb{R}^7$ yields one which is orthonormal with respect to the bilinear form determined by A. A bit more manipulation gives a matrix C with the property that $C^tAC = J$, where J is the diagonal matrix with (i,i)-entry equal to 1 for $i < 7$ and -1 for $i = 7$.

$$C = \begin{pmatrix} 1 & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & \frac{-1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{5}{3}} & -\sqrt{\frac{3}{5}} & & \\ 0 & 0 & 0 & 0 & \sqrt{\frac{2}{5}} & \frac{-1}{2}\sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{\frac{3}{2}} & \frac{-2}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

(That $C^tAC = J$ can easily be checked with a computer algebra system.) As in the proof of [19, Th. 7.2.4], for each i between 1 and 7 the ith column $\mathbf{v}_i$ of C is the inward-pointing normal to the face S_i of σ, which is itself the intersection with $\mathbb{H}^6$ of the image of the non-negative orthant $\{(x_1, \ldots, x_7) \,|\, x_i \geq 0\}$ under the inverse of the linear transformation determined by C^tJ.

For each i, the vertex $\mathbf{x}_i$ of σ opposite S_i is the intersection of the faces S_j for $j \neq i$. It is therefore characterized by the property that $\mathbf{x}_i \circ \mathbf{v}_j = 0$, $j \neq i$, where "$\circ$" refers to the Lorentzian inner product on $\mathbb{R}^7$. A little linear algebra therefore yields the following descriptions for the $\mathbf{x}_i$:

$$\begin{aligned} \mathbf{x}_7 &= (0,0,0,0,0,0,1) \\ \mathbf{x}_6 &= \left(0,0,0,0,0,\tfrac{-1}{\sqrt{3}},\tfrac{2}{\sqrt{3}}\right) \\ \mathbf{x}_5 &= \left(0,0,0,0,\tfrac{-1}{2},\tfrac{-1}{2}\sqrt{\tfrac{5}{3}},\sqrt{\tfrac{5}{3}}\right) \\ \mathbf{x}_4 &= \left(0,0,0,\tfrac{-1}{\sqrt{5}},-\sqrt{\tfrac{3}{10}},\tfrac{-1}{\sqrt{2}},\sqrt{2}\right) \\ \mathbf{x}_3 &= \left(0,0,\tfrac{-1}{\sqrt{2}},-\sqrt{\tfrac{3}{10}},\tfrac{-3}{2\sqrt{5}},\tfrac{-\sqrt{3}}{2},\sqrt{3}\right) \\ \mathbf{x}_2 &= \left(0,\tfrac{-1}{\sqrt{3}},0,\tfrac{-2}{\sqrt{15}},-\sqrt{\tfrac{2}{5}},-\sqrt{\tfrac{2}{3}},2\sqrt{\tfrac{2}{3}}\right) \\ \mathbf{x}_1 &= \left(-1,\tfrac{-1}{\sqrt{3}},0,\tfrac{-2}{\sqrt{15}},-\sqrt{\tfrac{2}{5}},-\sqrt{\tfrac{2}{3}},2\sqrt{\tfrac{2}{3}}\right)\cdot t \end{aligned}$$

Note that $\mathbf{x}_1$ depends on a real parameter t: this is because it does not lie in $\mathbb{H}^6$ but is a line in the light cone representing the sole ideal vertex of σ.

If a particular value of $t > 0$ is fixed, then a horoball of $\mathbb{H}^6$ centered at $\mathbf{x}_1$ consists of the set of points $\mathbf{y} \in \mathbb{H}^6$ satisfying $\mathbf{y} \circ \mathbf{x}_1 \geq -1$. (This perspective was exploited for instance by Epstein–Penner [9].) Direct computation shows that $\mathbf{x}_2 \circ \mathbf{x}_1 = -t$ is the largest value among the $\mathbf{x}_j \circ \mathbf{x}_1$, $2 \leq j \leq 7$, for any fixed $t > 0$. Therefore fixing $t = 1$ and calling the corresponding horoball B, we have that $\mathbf{x}_2$ lies on the boundary of B, with $\mathbf{x}_j$ outside B for all $j > 2$.

Below we summarize the development above, and some additional observations.

Lemma 3.1. *Let $\sigma \subset \mathbb{H}^6$ be the (generalized) hyperbolic simplex with Coxeter diagram given in Figure 1. For $1 \leq i \leq 7$, let S_i be the side of σ corresponding to the vertex labeled i in the figure, and let $\mathbf{x}_i$ be the vertex of σ opposite S_i.*

Among the $\mathbf{x}_i$, $1 \leq i \leq 7$, only $\mathbf{x}_1$ is ideal. Let B be the horoball of $\mathbb{H}^6$ centered at $\mathbf{x}_1$ which has $\mathbf{x}_2$ in its boundary. The totally geodesic hyperplane of $\mathbb{H}^6$ containing S_1 intersects B only at $\mathbf{x}_2$. In the Euclidean metric that ∂B inherits from $\mathbb{H}^6$,

$\sigma \cap \partial B$ is a simplex with volume $1/(2^{9.5} \cdot 5 \cdot 3)$. Finally, for $d_{\max} = \cosh^{-1}(\sqrt{3})$, the closed $d_{\max}$-neighborhood of $\mathbf{x}_7$ contains all of $\overline{\sigma - (\sigma \cap B)}$.

Proof. The subspace $V_1 = \{0\} \times \mathbb{R}^6$ of $\mathbb{R}^7$ intersects $\mathbb{H}^6$ in the totally geodesic hyperplane H_1 containing the face S_1 (and hence also $\mathbf{x}_2, \ldots, \mathbf{x}_7$ in particular): note that V_1 is clearly Lorentz-orthogonal to the first column $\mathbf{v}_1$ of the matrix C. For any $\mathbf{y} \in H_1$ we have

$$\mathbf{y} \circ \mathbf{x}_1 = \mathbf{y} \circ \mathbf{x}_2 \leq -1,$$

with equality if and only if $\mathbf{y} = \mathbf{x}_2$. Here the equality follows from the explicit descriptions of $\mathbf{x}_1$ and $\mathbf{x}_2$ and the fact that $\mathbf{y}$ has first entry equal to zero. The inequality above follows from a consequence of the Cauchy–Schwarz inequality: if $\mathbf{x} \circ \mathbf{x} = a \leq 0$ and $\mathbf{y} \circ \mathbf{y} = b \leq 0$, and $\mathbf{x}$ and $\mathbf{y}$ have positive nth entries, then $\mathbf{x} \circ \mathbf{y} \leq -\sqrt{ab}$, with equality if and only if $\mathbf{y}$ is a scalar multiple of $\mathbf{x}$. We thus find that $H_1 \cap B = \mathbf{x}_2$.

Let $\sigma' = \sigma \cap \partial B$, and for each $i > 1$ let $S_i' = S_i \cap \partial B$. Each such S_i' is a Euclidean hyperplane in the metric that ∂B inherits from $\mathbb{H}^6$, and the angle of intersection between S_i' and S_j' matches that of S_i and S_j. It follows that the Coxeter diagram of σ' is obtained from the one in Figure 1 by removing the vertex labeled 1 and the interior of the edge attached to it.

We now briefly recap the standard fact that σ' is the double of a simplex σ_0 which is a fundamental domain for the symmetries of a five-dimensional Euclidean cube. The cube is regular; that is, its symmetry group acts transitively on *flags*, tuples of the form $(F_0, F_1, F_2, F_3, F_4, F_5)$ where F_5 is the cube and F_i is a codimension-one face of F_{i+1} for each $i < 5$. For instance, taking $F_i = [0,1]^i \times \{\mathbf{0}_{5-i}\}$ for $0 \leq i \leq 5$ yields a flag of the cube $[0,1]^5$. (Here for any $j > 0$ and $r \in \mathbb{R}$, "$\mathbf{r}_j$" means the vector in $\mathbb{R}^j$ with all entries r.)

We associate a simplex to such a flag by placing a vertex at the *barycenter* of each F_i, the point fixed by all symmetries preserving F_i. Vertices associated to the sample flag above are of the form $\mathbf{y}_i = \frac{\mathbf{1}}{\mathbf{2}}_i \times \mathbf{0}_{5-i}$ for $0 \leq i \leq 5$. The cube is thus tiled by these simplices, which all have a vertex at its barycenter. The cube's symmetry group acts transitively on the simplices, so each is a copy of σ_0.

Below is the Coxeter diagram of the reflection group in the sides of σ_0:

This can be easily checked by an explicit calculation using the sample copy of σ_0 described above. From such a calculation one finds that the face T opposite the vertex F_0 corresponds to one of the endpoints of the diagram. That is, T is perpendicular to all other faces save one, which it intersects at an angle of $\pi/4$. Doubling σ_0 across T thus yields another simplex σ' which has four faces that are doubles of certain faces of σ_0 — those perpendicular to T. These faces have the same angles of intersection in σ' as in σ_0.

The remaining two faces of σ' are the face S_2' of σ_0 that meets T at an angle of $\pi/4$ and its image S_3' under reflection across T. These faces are thus perpendicular, and S_3' meets every other face at the same angle as S_2'. In particular they meet a common face S_4' at an angle of $\pi/3$ and all others at right angles. It follows that the Coxeter diagram of σ' is obtained from that of Figure 1 by removing the vertex labeled 1 and the interior of the edge attached to it, as claimed above. Moreover, the faces labeled S_2', S_3' and S_4' here play the same roles as the $S_i' = S_i \cap \partial B$ above.

This last observation can be combined with information about the vertices of our particular embedding of σ' to discern the edge lengths of the ambient cube. Note that the vertex of σ_0 opposite T is also the vertex of σ' opposite S_3', since T separates them. Similarly, the reflection of this vertex across T is opposite S_2' in σ'. And the vertex of σ_0 opposite T is F_0, a vertex of the ambient cube, whence also its reflected image is a vertex of the cube, and the two vertices share an edge.

On the other hand, in our embedding of σ', its vertices opposite S_2' and S_3' are the orthogonal projections $\mathbf{x}_2'$ and $\mathbf{x}_3'$ of $\mathbf{x}_2$ and $\mathbf{x}_3$, respectively, to ∂B. Since $\mathbf{x}_2 \in \partial B$ we have $\mathbf{x}_2' = \mathbf{x}_2$. The projection of $\mathbf{x}_3$ to ∂B is along the geodesic ray

$$\gamma(t) = e^{-t}\mathbf{x}_3 - \left(\frac{\sinh t}{\mathbf{x}_3 \circ \mathbf{x}_1}\right)\mathbf{x}_1, \quad t \geq 0.$$

(One can verify directly that this is a geodesic ray in $\mathbb{H}^6$, parametrized by arclength, that starts at $\mathbf{x}_3$ and projectively approaches the class of $\mathbf{x}_1$ as $t \to \infty$.) Its intersection with ∂B occurs at $t = \log(-\mathbf{x}_3 \circ \mathbf{x}_1) = \log\sqrt{2}$, so $\mathbf{x}_3' = \gamma(\log\sqrt{2}) = \frac{1}{\sqrt{2}}\mathbf{x}_3 + \frac{1}{4}\mathbf{x}_1$. The hyperbolic distance d from $\mathbf{x}_2'$ to $\mathbf{x}_3'$ satisfies

$$\cosh d = -\mathbf{x}_2' \circ \mathbf{x}_3' = -\frac{1}{\sqrt{2}}\mathbf{x}_2 \circ \mathbf{x}_3 - \frac{1}{4}\mathbf{x}_2 \circ \mathbf{x}_1 = \frac{5}{4},$$

since $\mathbf{x}_2 \circ \mathbf{x}_3 = -\sqrt{2}$ and $\mathbf{x}_2 \circ \mathbf{x}_1 = 1$. Using the fact that the Euclidean distance ℓ from $\mathbf{x}_2'$ to $\mathbf{x}_3'$ in ∂B satisfies $\ell/2 = \sinh(d/2)$ we obtain $\ell = 1/\sqrt{2}$. This is thus the sidelength of the ambient Euclidean cube.

Because an n-dimensional cube has $2n$ faces, the 5-cube has $2^5 \cdot 5! = 2^8 \cdot 5 \cdot 3$ flags, so it is tiled by this is the number of copies of σ_0. Since σ' is the double of σ_0, the ratio of its volume to that of the ambient cube is 1 to $2^7 \cdot 5 \cdot 3$. And since the cube itself has edgelength $1/\sqrt{2}$ and therefore Euclidean volume $1/2^{2.5}$ we obtain the claimed volume for σ'.

We finally address the claim regarding $d_{\max} = \cosh^{-1}(\sqrt{3})$. Suppose $\mathbf{p} = \sum_{i=1}^7 t_i\mathbf{x}_i$ is an element of σ outside the interior of B, so

$$\mathbf{p} \circ \mathbf{x}_1 = \sum_{i=2}^7 t_i\mathbf{x}_i \circ \mathbf{x}_1 \leq -1 \quad \Rightarrow \quad \sum_{i=2}^7 t_i(-\mathbf{x}_i \circ \mathbf{x}_1) \geq 1.$$

(Recall that $\mathbf{x}_i \circ \mathbf{x}_1 < 0$ for each i, since $\mathbf{x}_i \circ \mathbf{x}_i = -1$ and $\mathbf{x}_1 \circ \mathbf{x}_1 = 0$.) That $\mathbf{p}$ lies in σ means $t_i \geq 0$ for all i and

$$\mathbf{p} \circ \mathbf{p} = \mathbf{p}_0 \circ \mathbf{p}_0 + 2t_1\sum_{i=2}^7 t_i\mathbf{x}_i \circ \mathbf{x}_1 = -1.$$

Here $\mathbf{p}_0 = \sum_{i=2}^7 t_i\mathbf{x}_i$. Solving for t_1 yields

$$t_1 = \frac{-1 - \mathbf{p}_0 \circ \mathbf{p}_0}{2\sum_{i=2}^7 t_i\mathbf{x}_i \circ \mathbf{x}_1} \leq \frac{1}{2}(1 + \mathbf{p}_0 \circ \mathbf{p}_0).$$

Note that since t_1 is non-negative we must have $\mathbf{p}_0 \circ \mathbf{p}_0 \geq -1$. We now observe that for each i, the inner product $\mathbf{x}_7 \circ \mathbf{x}_i$ is the opposite of the final entry of $\mathbf{x}_i$. The least of these quantities is $\mathbf{x}_7 \circ \mathbf{x}_3 = -\sqrt{3}$. So we immediately obtain the inequality $\mathbf{p} \circ \mathbf{x}_7 \geq -\sqrt{3}\sum_{i=1}^7 t_i$. Since $\mathbf{x}_i \circ \mathbf{x}_i = -1$ for each $i > 1$, we have

$$\mathbf{p}_0 \circ \mathbf{p}_0 = -\sum_{i=2}^7 t_i^2 + 2\sum_{i \neq j} t_it_j\mathbf{x}_i \circ \mathbf{x}_j \leq -\left(\sum_{i=2}^7 t_i\right)^2.$$

Therefore $\mathbf{p}\circ\mathbf{x}_7 \geq -\sqrt{3}(t_1+\sqrt{-\mathbf{p}_0\circ\mathbf{p}_0}) \geq -\frac{\sqrt{3}}{2}(1+2\sqrt{-\mathbf{p}_0\circ\mathbf{p}_0}+\mathbf{p}_0\circ\mathbf{p}_0)$. A calculus argument shows that this is at least $\sqrt{-3}$ regardless of the value of $\mathbf{p}_0\circ\mathbf{p}_0$ in $[-1,0]$. This proves that $d(\mathbf{p},\mathbf{x}_7)\leq d_{\max}$, since their distance is defined as the inverse hyperbolic cosine of $-\mathbf{p}\circ\mathbf{x}_7$. □

Corollary 3.2. *Let $\sigma\subset\mathbb{H}^6$ be the generalized hyperbolic simplex with Coxeter diagram given in Figure 1, and let G be the group generated by reflections in the sides of σ corresponding to vertices 1 through 6. Then $P=\bigcup\{g(\sigma)\,|\,g\in G\}$ is a right-angled polyhedron of finite volume, and for B as in Lemma 3.1, $\mathcal{B}=\{g(B)\,|\,g\in G\}$ is a collection of horoballs that are embedded in the sense of Definition 1 and pairwise non-overlapping, with one for each ideal vertex of P. For $d_{\max}$ as in Lemma 3.1, $\overline{P-\bigcup\{B\in\mathcal{B}\}}$ is contained in the closed ball of radius $d_{\max}$ about $\mathbf{x}_7$, and it has volume*

$$2^7\cdot 3^4\cdot 5\left(\frac{\pi^3}{2^7\cdot 5^2\cdot 3^5}-\frac{1}{2^{9.5}\cdot 5^2\cdot 3}\right)=\frac{2^{2.5}\pi^3-3^4}{2^{2.5}\cdot 5\cdot 3}\approx 1.112.$$

Proof. That P is a right-angled polyhedron follows from the fact that its face S_7 corresponding to vertex 7 intersects every other face at an angle of $\pi/2$ or $\pi/4$, see [1, Lemma 3.4]. Let H_0 be the subgroup of G generated by reflections in the faces S_2 through S_6 of σ, and let $P_0=\bigcup\{h(\sigma)\,|\,h\in H_0\}$. Then P is a non-overlapping union of translates of P_0, one for each (say, left) coset of H_0. By construction, each side of P_0 is either a union of H_0-translates of S_1 or of S_7. The sides of the former kind comprise the frontier of P_0 in P; those of the latter lie in the frontier of P.

We claim that $P\cap B=P_0\cap B$. By Lemma 3.1, B is entirely contained in the half-space bounded by the geodesic hyperplane $\mathcal{H}_1$ containing S_1 that also contains σ. Since each of S_2 through S_6 contains the ideal vertex $\mathbf{x}_1$, H_0 stabilizes B, so each H_0-translate of $\mathcal{H}_1$ also bounds a half-space containing both B and the corresponding translate of σ. If $\mathcal{H}_1,\ldots,\mathcal{H}_n$ is the list of such translates containing a side of P_0, it follows that both B and P_0 are contained in an intersection of half-spaces bounded by the $\mathcal{H}_i$. Therefore since the frontier of P_0 in P is a union of H_0-translates of S_1, each point of $P-P_0$ is separated from P_0 by some $\mathcal{H}_i$. This proves the claim.

The claim implies that $\mathcal{B}=\{g(B)\,|\,g\in G\}$ is embedded and pairwise non-overlapping: $\mathcal{B}$ corresponds bijectively to the set of cosets of H_0 in G, and the intersection of each element with P is contained in a corresponding translate of P_0.

The remaining claims follow from the fact that $\overline{P-\bigcup\{B\in\mathcal{B}\}}$ is a union of G-translates of $\overline{\sigma-(\sigma\cap B)}$, where G is a group of isometries fixing $\mathbf{x}_7$. This and the final claim of Lemma 3.1 immediately imply that $\overline{P-\bigcup\{B\in\mathcal{B}\}}$ is contained in the ball of radius $d_{\max}$ about $\mathbf{x}_7$. For the volume, we appeal to [11], which asserts that σ has volume $\pi^3/777,600$ (see p. 344 there). The volume of $\sigma'=\sigma\cap\partial B$ is recorded in Lemma 3.1, and the volume of $\sigma\cap B$ is one-fifth this quantity. (This follows from a general fact that can be proven using horoballs centered at infinity in the upper half-space model $\{(x_1,\ldots,x_n)\,|\,x_n>0\}$ for $\mathbb{H}^n$, where the hyperbolic volume form is the Euclidean volume form scaled by $1/x_n^n$.) Subtracting one from the other, and multiplying the result by the order of G, gives the formula claimed. □

Corollary 3.3. *Suppose $M=\mathbb{H}^3/\Gamma$ is a closed arithmetic hyperbolic 3-manifold such that Γ is commensurable with $\mathrm{SO}(q,\mathbb{Z})$ for some $\mathbb{Q}$-defined form q. For any $\alpha\in\Gamma-\{1\}$, there exists a subgroup H' of $\pi_1 M$ such that $\alpha\notin H'$, and the index of*

H' is bounded above by

$$2^7 3^4 5 \cdot C_1 x^{21} \mathrm{vol}(M) \cdot \frac{v_5(1)}{V_0} \sinh^5\left(2(2R + d_{\max} + \log p^{-1}(\mathrm{vol}(M)))\right) \ell(\alpha),$$

where $v_5(1) = 8\pi^2/15$ and:

- *$\ell(\alpha)$ is the length of the unique geodesic representative of α;*
- *$R = \ln(\sqrt{6} + \sqrt{7})$;*
- *$C_1 = 2^{2\log(3)/\log(3/2)}/\log(3/2)$, see Lemma 2.1;*
- *$x = x(q)$ is as described in the formula (5);*
- *$V_0 = \frac{2^{2.5}\pi^3 - 3^4}{2^{2.5}.5 \cdot 3} \approx 1.112$ and $d_{\max} = \cosh^{-1}(\sqrt{3})$, see Corollary 3.2; and*
- *$p(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x - \frac{8}{15}$.*

Proof. Fix $\alpha \in \Gamma - \{1\}$. By Theorem 2.3, Γ has a subgroup Δ that injects to $\mathrm{SO}(6,1;\mathbb{Z})$, with index at most $C_1 D\mathrm{vol}(M)$, for C_1 and D as described in Lemmas 2.1 and 2.2, respectively. By the discussion above, $D \leq x^{21}$ for x as in (5). So Δ has index at most $C_1 x^{21}\mathrm{vol}(M)$, and if $\alpha \notin \Delta$ then we are done. So we now assume that it is.

Since P is the union of $2^7 3^4 5$ copies of σ, the reflection group Γ_P in its sides has that index in the reflection group $\mathrm{SO}(6,1;\mathbb{Z})$ in the sides of σ. Therefore $\Gamma_P \cap \Delta$ has index at most $2^7 3^4 5 \cdot C_1 x^{21}\mathrm{vol}(M)$ in Γ. If $\alpha \in \Delta - \{1\} \subset \mathrm{SO}(6,1;\mathbb{Z})$ is not in Γ_P then again we are done, so we now suppose that $\alpha \subset \Gamma_P$. We will finally apply Theorem 1.6 to obtain the stated bound.

The remaining constants in the Corollary's statement are obtained by specializing those of Theorem 1.6 to our example. For instance, the general formula $v_n(1) = \pi^{n/2}/\Gamma(\frac{n}{2}+1)$ takes the value $8\pi^2/15$ when $n+1 = 6$. And the volume V_0 of $\overline{P - \bigcup\{B \in \mathcal{B}\}}$ is less than $V_{R+h_{\max}}$.

The polynomial p above arises from the computation of the volume $V_6(r)$ of a ball in $\mathbb{H}^6$ of radius r:

$$V_6(r) = \pi^3 \int_0^r \sinh^5 t\, dt = \pi^3 p(\cosh r).$$

This is used to bound $h_{\max}$ above in terms of the volume of M. Corollary 1.3 implies that $h_{\max}$ is at most $\log\cosh R$, where R is the radius of the largest ball embedded in M. For $p(x)$ as above we have $\cosh R \leq p^{-1}(\mathrm{vol}(M))$, so $h_{\max} \leq \log p^{-1}(\mathrm{vol}(M))$. Since a ball about $\mathbf{x}_7$ of radius $d_{\max}$ contains all of $P - \bigcup\{B \in \mathcal{B}\}$ by Corollary 3.2, we may bound $d_{R+h_{\max}}$ by twice the radius $R + d_{\max} + h_{\max}$ of a ball at v. □

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