Time-Series-Cross-Section Data Analysis

Multiple Source Spatial and Spatio-Temporal Models

Jude C. Hays
jch61@pitt.edu

March 10, 2021

Outline

A Typology of Spatial Models

Table: Spatial Econometric Models

Name	Structural Model	Restrictions
General Nesting Model	$y = \rho Wy + X\beta + WX\theta + u, u = \lambda Wu + \epsilon$	none
Spatial Durbin Error Model	$y = X\beta + WX\theta + u, u = \lambda Wu + \epsilon$	$\rho = 0$
Spatial Autocorrelation Model	$y = \rho Wy + X\beta + u, u = \lambda Wu + \epsilon$	$\theta = 0$
Spatial Durbin Model	$y = \rho Wy + X\beta + WX\theta + \epsilon$	$\lambda = 0$
Spatial Autoregressive	$y = \rho Wy + X\beta + \epsilon$	$\lambda = \theta = 0$
Spatially Lagged X's	$y = X\beta + WX\theta + \epsilon$	$\rho = \lambda = 0$
Spatial Error Model	$y = X\beta + u, u = \lambda Wu + \epsilon$	$\rho = \theta = 0; \lambda = -\rho\beta$
(Non-Spatial) Linear Model	$y = X\beta + \epsilon$	$\rho = \lambda = \theta = 0$

Are these models identified?

The General Nesting Spatial Model (**GNS**):

$$\mathbf{y} = \rho\mathbf{W}\mathbf{y} + \mathbf{X}\beta + \mathbf{W}\mathbf{X}\theta + \mathbf{u}, \text{ where } \mathbf{u} = \lambda\mathbf{W}\mathbf{u} + \boldsymbol{\epsilon} \qquad (1)$$

After some algebraic manipulation the **GNS** model given in 1 can be re-written as:

$$\mathbf{y} = (\rho + \lambda)\mathbf{W}\mathbf{y} - \rho\lambda\mathbf{W}^2\mathbf{y} + \mathbf{X}\beta + (\theta - \lambda\beta)\mathbf{W}\mathbf{X} - \lambda\theta\mathbf{W}^2\mathbf{X} + \boldsymbol{\epsilon} \qquad (2a)$$

$$\mathbf{y} = q_1\mathbf{W}\mathbf{y} - q_2\mathbf{W}^2\mathbf{y} + \mathbf{X}q_3 + q_4\mathbf{W}\mathbf{X} - q_5\mathbf{W}^2\mathbf{X} + \boldsymbol{\epsilon} \qquad (2b)$$

Are these models identified?

The reduced-form of the **GNS** provides five parameters from which we can recover the four structural parameters. Substituting q_1 into q_2 and q_4 into q_5 gives a set of quadratic relationships for λ

$$\mathbf{y} = (\rho + \lambda)\mathbf{W}\mathbf{y} - \rho\lambda\mathbf{W}^2\mathbf{y} + \mathbf{X}\beta + (\theta - \lambda\beta)\mathbf{W}\mathbf{X} - \lambda\theta\mathbf{W}^2\mathbf{X} + \boldsymbol{\epsilon}$$
$$\mathbf{y} = q_1\mathbf{W}\mathbf{y} - q_2\mathbf{W}^2\mathbf{y} + \mathbf{X}q_3 + q_4\mathbf{W}\mathbf{X} - q_5\mathbf{W}^2\mathbf{X} + \boldsymbol{\epsilon}$$

$$\lambda^2 = q_2 + \lambda q_1$$
$$\lambda^2 = \frac{-(q_5 + \lambda q_4)}{\beta}$$

These equations provide a unique solution for λ and, in turn, the other parameters:

$$\lambda = \frac{-(\beta q_2 + q_5)}{\beta q_1 + q_4} \qquad \rho = q_1 + \frac{(\beta q_2 + q_5)}{\beta q_1 + q_4} \qquad \theta = q_4 + \frac{\beta(\beta q_2 + q_5)}{\beta q_1 + q_4}.$$

Non-Spatial DGP

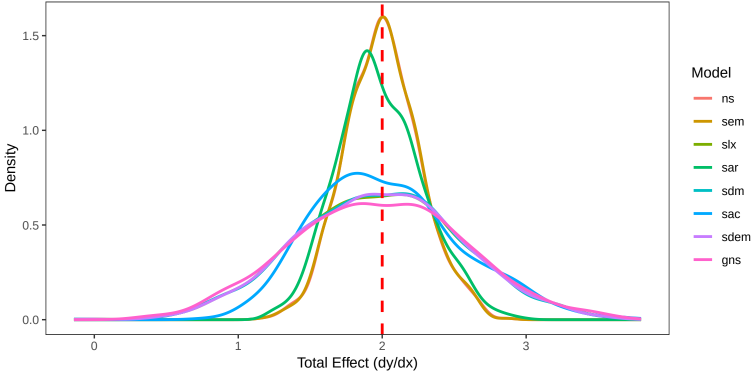


Figure: Total Effect Estimates, NS DGP (true value is dashed vertical line)

Spatial Error DGP

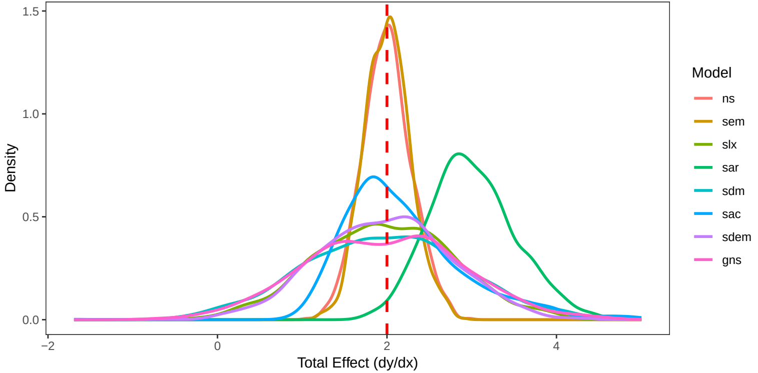


Figure: Total Effect Estimates, SEM DGP (true value is dashed vertical line)

Spatially-Lagged X DGP

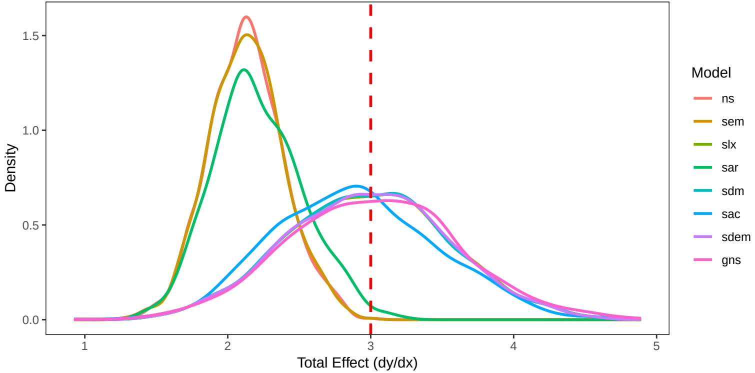


Figure: Total Effect Estimates, SLX DGP (true value is dashed vertical line)

Spatial Autoregressive DGP

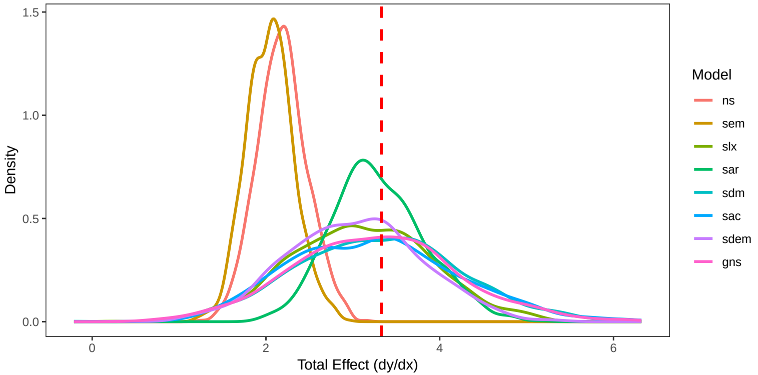


Figure: Total Effect Estimates, SAR DGP (true value is dashed vertical line)

Spatial Durbin DGP

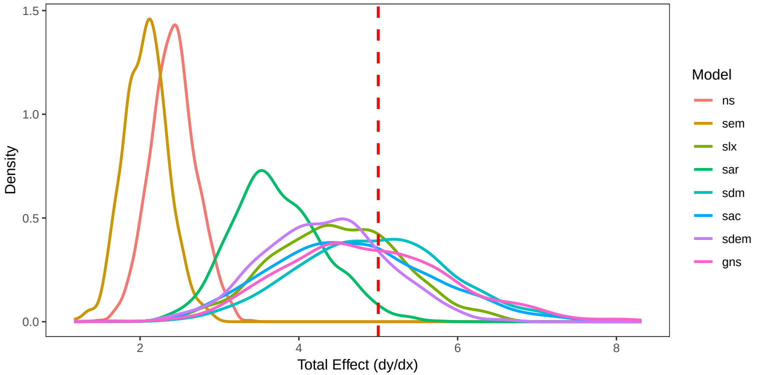


Figure: Total Effect Estimates, SDM DGP (true value is dashed vertical line)

Spatial Autocorrelation DGP

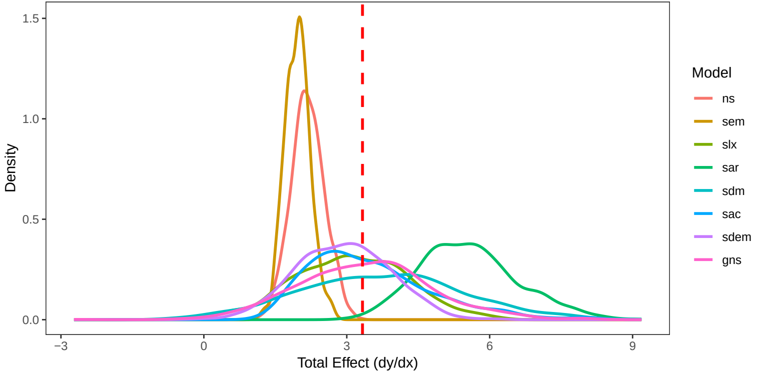


Figure: Total Effect Estimates, SAC DGP (true value is dashed vertical line)

Spatial Durbin Error DGP

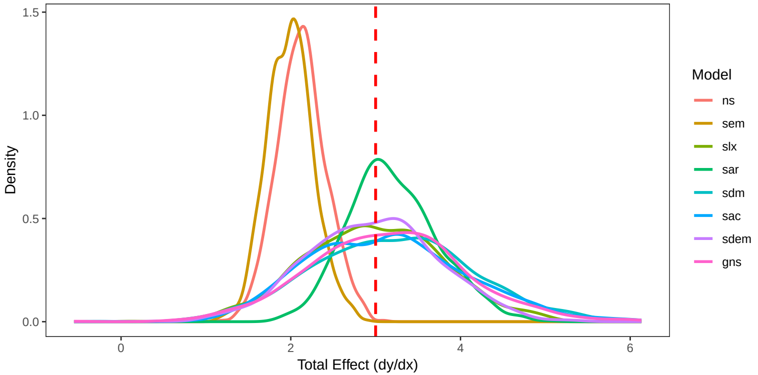


Figure: Total Effect Estimates, SDEM DGP (true value is dashed vertical line)

General Nesting DGP

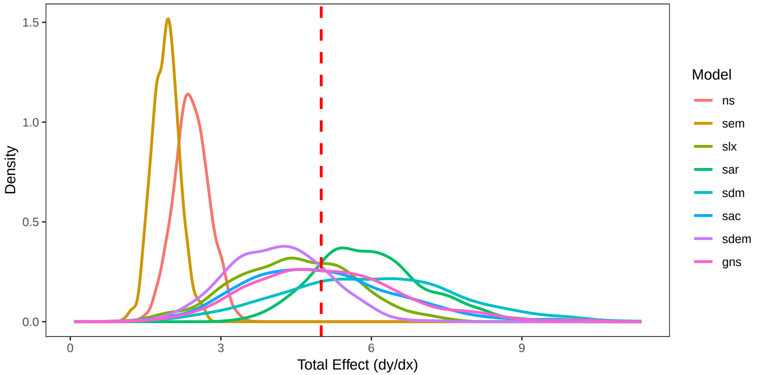


Figure: Total Effect Estimates, GNS DGP (true value is dashed vertical line)

The General STADL Model

The general version of the STADL$(ty^p, tx^q, te^r, sy^P, sx^Q, sx^R)$ is

$$\mathbf{M}\mathbf{y}_t = \mathbf{F}\mathbf{x}_t + \mathbf{A}\boldsymbol{\varepsilon}_t, \tag{4a}$$

$$\mathbf{M} \equiv \left(\mathbf{I} - \phi_1\mathbf{L} - \ldots - \phi_p\mathbf{L}^p - \rho_0\mathbf{W} - \ldots - \rho_{P-1}\mathbf{W}^{P-1}\right), \tag{4b}$$

$$\mathbf{F} \equiv \left(\mathbf{I}\beta + \mathbf{L}\gamma_1 + \ldots + \mathbf{L}^q\gamma_q + \mathbf{W}\theta_0 + \ldots + \mathbf{W}^{Q-1}\theta_{Q-1}\right), \tag{4c}$$

$$\mathbf{A} \equiv \left(\mathbf{I} - \delta_1\mathbf{L} - \ldots - \delta_r\mathbf{L}^r - \lambda_0\mathbf{W} - \ldots - \lambda_{R-1}\mathbf{W}^{R-1}\right)^{-1}. \tag{4d}$$

The First-Order STADL Model

We express a first-order STADL conveniently for interpretation of spatiotemporal effects as:

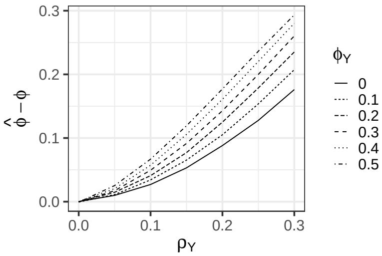
$$\mathbf{y} = \phi\mathbf{L}\mathbf{y} + \rho\mathbf{W}\mathbf{y} + \mathbf{x}\beta + \mathbf{L}\mathbf{x}\gamma + \mathbf{W}\mathbf{x}\theta + (\mathbf{I} - \delta\mathbf{L} - \lambda\mathbf{W})^{-1}\boldsymbol{\varepsilon}, \quad (5a)$$

$$\mathbf{y} = (\mathbf{I} - \phi\mathbf{L} - \rho\mathbf{W})^{-1}\left(\mathbf{x}\beta + \mathbf{L}\mathbf{x}\gamma + \mathbf{W}\mathbf{x}\theta + (\mathbf{I} - \delta\mathbf{L} - \lambda\mathbf{W})^{-1}\boldsymbol{\varepsilon}\right). \quad (5b)$$

Monte Carlos

To explore SAR or LDV estimation performance given dependence also in the unmodeled (or, implicitly, mismodeled) other dimension, we generate data from a STADL(sy^0, ty^1), i.e., the first-order spatiotemporal autoregressive model:

$$\mathbf{y}_t = \phi_y \mathbf{y}_{t-1} + \rho_y \mathbf{W}\mathbf{y}_t + \mathbf{x}_t \beta + \boldsymbol{\varepsilon}_{\mathbf{y}},$$
$$\mathbf{x}_t = \phi_x \mathbf{x}_{t-1} + \rho_x \mathbf{W}\mathbf{x}_t + \boldsymbol{\varepsilon}_{\mathbf{x}},$$



Monte Carlos

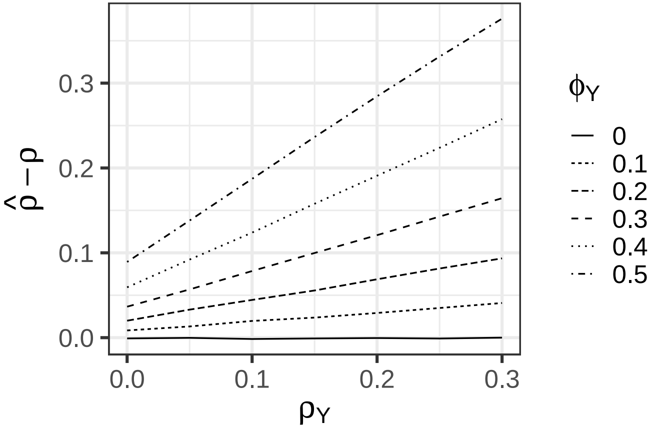


Figure: SAR Performance with Temporal Dependence — Bias in ρ_y