Time-Series Cross-Section Analysis

Enders, Chapter 1: Difference Equations

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February 3, 2021

Outline

Stochastic Difference Equations

"A difference equation expresses the value of a variable as a function of its own lagged values, time, and other variables...The reason for introducing...[these] equations is to make the point that time-series econometrics is concerned with the estimation of difference equations containing stochastic components" (Enders, p.3).

- 1^{st}, 2^{nd}, and n^{th} difference Δ, Δ^2, and Δ^n
- 1^{st}, 2^{nd}, and n^{th} order linear difference equations

$$y_t = a_0 + a_1 y_{t-1} + x_t$$
$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + x_t$$
$$y_t = a_0 + \sum_{i=1}^{n} a_i y_{t-i} + x_t$$

Theory Evaluation

Many economic theories generate model specifications in the form of difference equations

- The Random Walk Hypothesis
- Reduced-form and Structural Equations
- Error-Correction: Forward and Spot Prices
- Non-linear dynamics

Paper stones is an example of political theory that generates a model specification in the form of difference equations

Solving by Iteration

"A solution to a difference equation expresses the value of y_t as a function of the elements of the x_t sequence and t ... and possibly initial conditions" (Enders, p.9).

$$
\begin{aligned}
y_t &= a_0 + a_1 y_{t-1} + \varepsilon_t \\
y_1 &= a_0 + a_1 y_0 + \varepsilon_1 \\
y_2 &= a_0 + a_1 [a_0 + a_1 y_0 + \varepsilon_1] + \varepsilon_2 \\
y_2 &= a_0 + a_1 a_0 + (a_1)^2 y_0 + a_1 \varepsilon_1 + \varepsilon_2 \\
&\vdots \\
y_t &= a_0 \sum_{i=0}^{t-1} (a_1)^i + (a_1)^t y_0 + \sum_{i=0}^{t-1} (a_1)^i \varepsilon_{t-i}
\end{aligned}
$$

The Dynamics of First-Order Difference Equations

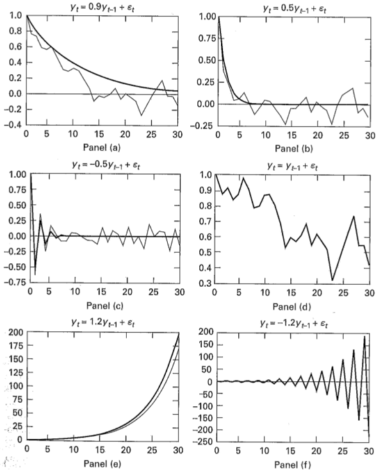


FIGURE 1.2 Convergent and Nonconvergent Sequences

Solving n^{th} Order Difference Equations

The complete n^{th} order difference equation is

$$y_t = a_0 + \sum_{i=1}^{n} a_i y_{t-i} + x_t$$

The homogeneous portion of the n^{th} order difference equation is

$$y_t = \sum_{i=1}^{n} a_i y_{t-i}$$

- A homogeneous solution to an n^{th} order difference equation is a solution to the homogeneous portion of the difference equation. There should be n solutions.
- A particular solution is a solution to the original complete difference equation.
- A general solution to an n^{th} order difference equation is a particular solution plus all homogeneous solutions.

The Solution Methodology

1. Form the homogeneous equation and find all n homogeneous solutions;
2. Find a particular solution;
3. Obtain the general solution as the sum of the particular solution and a linear combination of all homogeneous solutions;
4. Eliminate the arbitrary constant(s) by imposing the initial condition(s) on the general solution.

Solving Homogeneous Difference Equations

Ex. 1 (First-order): $y_t = .9y_{t-1}$

- The homogeneous solution will take the form $y_t^h = A\alpha^t$
- The goal is to solve for A and α
- Substitute for y_t

$$A\alpha^t - .9A\alpha^{t-1} = 0$$

- Divide by $A\alpha^{t-1}$

$$\alpha - .9 = 0$$

- So, now we have

$$y_t^h = A(.9)^t$$

Solving Homogeneous Difference Equations

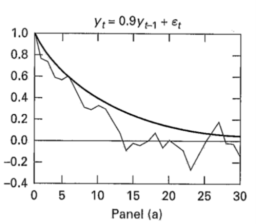
Ex. 1 (First-order): $y_t = .9y_{t-1}$

- We can eliminate the arbitrary constant if we know the outcome in the initial period y_0

$$y_0 = A(.9)^0$$

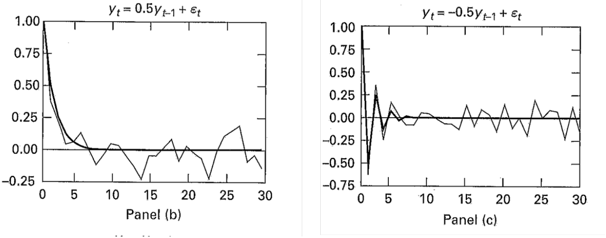
- If we set $y_0 = 1$, we have our final solution

$$y_t^h = (.9)^t$$



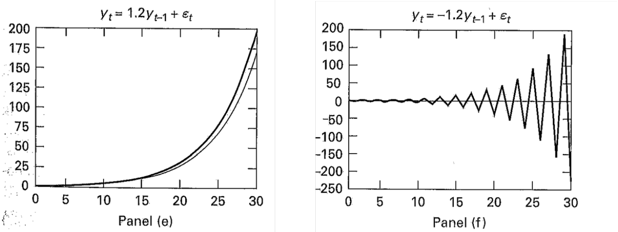
Stability Conditions for First-Order Solutions

If $|\alpha| < 1$, then α^t converges to zero as t goes to infinity. Convergence is direct if $0 < \alpha < 1$ and oscillatory if $-1 < \alpha < 0$.



Stability Conditions for First-Order Solutions

If $|\alpha| > 1$, the solution is not stable. If $\alpha > 1$, then α^t converges to infinity as t goes to infinity. If $\alpha < 1$ and the solution oscillates explosively.



Solving Homogeneous Difference Equations

Ex. 2 (Second-order): $y_t = 3 + .9y_{t-1} - .2y_{t-2}$

- Again, the homogeneous solutions will take the form $y_t^h = A\alpha^t$
- The goal is to solve for A_1, A_2, α_1, and α_2
- Substitute for y_t

$$A\alpha^t - .9A\alpha^{t-1} + .2A\alpha^{t-2} = 0$$

- Divide by $A\alpha^{t-2}$

$$\alpha^2 - .9\alpha + .2 = 0$$

- There are two solutions. We solve for α_1 and α_2 using the quadratic formula

$$\alpha_1, \alpha_2 = \frac{.9 \pm \sqrt{.81 - 4(.2)}}{2} = .5, .4$$

Solving Homogeneous Difference Equations

Ex. 2 (Second-order): $y_t = 3 + .9y_{t-1} - .2y_{t-2}$

- So, now we have

$$y_t = A_1(.5)^t + A_2(.4)^t$$

- We can eliminate the arbitrary constants if we know the outcome in the initial periods y_0 and y_1
- If we set $y_0 = 13$ and $y_1 = 11.3$, for instance, we two equations and two unknowns. To these equations, we need to add the particular (steady-state) solution $(c = 3/(1 - .9 + .2))$.

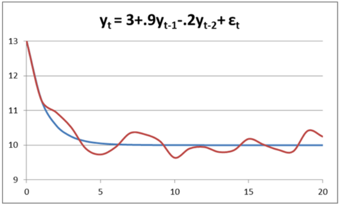
$$13 = 10 + A_1 + A_2$$
$$11.3 = 10 + A_1(.5) + A_2(.4)$$

Solving Homogeneous Difference Equations

Ex. 2 (Second-order): $y_t = 3 + .9y_{t-1} - .2y_{t-2}$

- Solving gives us $A_1 = 1$ and $A_2 = 2$, and our final solution is

$$y_t = (.5)^t + 2(.4)^t$$



$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Stability Conditions for Second-Order Solutions

stability requires as all characteristic roots lie within the unit circle (Enders, p. 29).

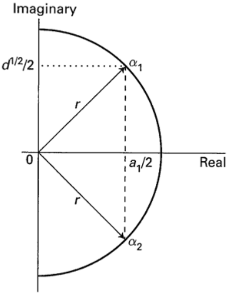


FIGURE 1.6 Characteristic Roots and the Unit Circle

$r < 1$

$$\alpha = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Stability Conditions for Higher-Order Systems

Higher-Order Systems: $y_t - \sum_{i=1}^{n} a_i y_{t-i} = 0$

Oftentimes, we do not need to solve for the characteristic roots of higher-order systems.

1. A necessary condition for stability is $\sum_{i=1}^{n} a_i < 1$
2. A sufficient condition for stability is $\sum_{i=1}^{n} |a_i| < 1$
3. The process contains a unit root if $\sum_{i=1}^{n} a_i = 1$

Particular Solutions for Deterministic Processes

If $\mathbf{x_t} = \mathbf{0}$, the difference equation becomes

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \ldots + a_n y_{t-n},$$

which is solved when $\Delta y_t = 0$ or $y_t = y_{t-1} = y_{t-2} = y_{t-n} = c$.

- Substituting for y_t gives

$$c = a_0 + a_1 c + a_2 c + \ldots + a_n c$$

- Solving for c gives

$$c = a_0/(1 - a_1 - a_2 - \ldots - a_n)$$

- Thus, a particular solution is

$$y_t = a_0/(1 - a_1 - a_2 - \ldots - a_n)$$

Particular Solutions for Stochastic Processes

The Method of Undetermined Coefficients

1. Since linear equations have linear solutions, we know the form of the solution.
2. Posit a linear *challenge solution* that includes all the terms thought to appear in the solution.
3. Solve for the undetermined coefficients.

The Method of Undetermined Coefficients

Ex. 3 (First-Order): $y_t = 3 + .9y_{t-1} + \varepsilon_t$

- Posit a linear *challenge solution* for the stochastic portion of the particular solution

$$y_t = \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i}$$

- Substitute the challenge solution into the difference equation

$$\alpha_0\varepsilon_t + \alpha_1\varepsilon_{t-1} + \alpha_2\varepsilon_{t-2} + \ldots = .9[\alpha_0\varepsilon_{t-1} + \alpha_1\varepsilon_{t-2} + \alpha_2\varepsilon_{t-3} + \ldots] + \varepsilon_t$$

The Method of Undetermined Coefficients

Ex. 3 (First-Order): $y_t = 3 + .9y_{t-1} + \varepsilon_t$

- Collect like terms

$$(\alpha_0 - 1)\varepsilon_t + (\alpha_1 - .9\alpha_0)\varepsilon_{t-1} + (\alpha_2 - .9\alpha_1)\varepsilon_{t-2} + ... = 0$$

- Verify that there are coefficient values that make the challenge solution a solution for the difference equation.

$$\begin{aligned}(\alpha_0 - 1) &= 0\\(\alpha_1 - .9\alpha_0) &= 0\\(\alpha_2 - .9\alpha_1) &= 0\\&\vdots\end{aligned}$$

- Solving for α_i, we have $\alpha_i = (.9)^i$

Putting it all together

Ex. 3 (First-Order): $y_t = 3 + .9y_{t-1} + \varepsilon_t$

- This gives the general solution

$$y_t = 30 + A(.9)^t + \sum_{i=0}^{\infty} (.9)^i \varepsilon_{t-i}$$

- We can eliminate the arbitrary constant if we have an initial value for y_0.

$$y_0 = 30 + A + \sum_{i=0}^{\infty} (.9)^i \varepsilon_{-i}$$

Putting it all together

Ex. 3 (First-Order): $y_t = 3 + .9y_{t-1} + \varepsilon_t$

- Substituting A into the general solution gives

$$y_t = 30 + \left[y_0 - 30 - \sum_{i=0}^{\infty} (.9)^i \varepsilon_{-i} \right] (.9)^t + \sum_{i=0}^{\infty} (.9)^i \varepsilon_{t-i}$$

- Collecting like terms, we have

$$y_t = 30 + (.9)^t [y_0 - 30] + \sum_{i=0}^{\infty} (.9)^i \varepsilon_{t-i} - (.9)^t \sum_{i=0}^{\infty} (.9)^i \varepsilon_{-i}$$

Putting it all together

Ex. 3 (First-Order): $y_t = 3 + .9y_{t-1} + \varepsilon_t$

- The stochastic portion of this solution can be simplified. To see this, write out the case for $t = 1$.

$$\varepsilon_1 + (.9)\varepsilon_0 + (.9)^2\varepsilon_{-1} + \ldots - (.9)[\varepsilon_0 + (.9)\varepsilon_{-1} + \ldots]$$

- Thus, we have

$$y_t = 30 + (.9)^t\,[y_0 - 30] + \sum_{i=0}^{t-1} (.9)^i \varepsilon_{t-i}$$

Lag Operators and their Properties

$$L^i y_t \equiv y_{t-i}$$

1. The lag of a constant is a constant: $Lc = c$.
2. The **distributive law** holds:
   $(L^i + L^j)y_t = L^i y_t + L^j y_t = y_{t-i} + y_{t-j}$.
3. The **associative law** holds:
   $L^i L^j y_t = L^i(L^j y_t) = L^i y_{t-j} = y_{t-i-j}$.
4. L raised to a negative number is the lead operator:
   $L^{-i} y_t = y_{t+i}$.
5. For $|a| < 1$ the infinite sum $(1 + aL + a^2L^2 + a^3L^3 + ...)y_t$ converges to $y_t/(1 - aL)$.

Lag Operators and their Properties

$$L^i y_t \equiv y_{t-i}$$

- Lag Operators allow us to write high-order difference equations,

$$(1 - a_1 L - a_2 L^2 + ... a_p L^p) y_t = a_0 + (1 + b_1 L - b_2 L^2 + ... b_q L^q) \varepsilon_t$$
$$A(L) y_t = a_0 + B(L) \varepsilon_t$$

as well as their particular solutions, compactly:

$$y_t = a_0 / A(L) + B(L) \varepsilon_t / A(L).$$