Discrete Green’s functions and random walks on graphs

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Abstract

We prove an explicit formula of Chung–Yau’s Discrete Green’s functions as well as hitting times of random walks on graphs. The formula is expressed in terms of two natural counting invariants of graphs. Uniform derivations of Green’s functions and hitting times for trees and other special graphs are given.

1. Introduction

Let \( \Gamma = (V, E) \) be a simple graphs (without loops or multi-edges) with \(|V| = n\) vertices. The \textit{volume} of a graph is \( \text{vol}(\Gamma) = \sum_{v \in V} d_v \), where \( d_v \) is the valence of \( v \) in \( \Gamma \).

The \textit{Laplacian} of \( \Gamma \) is the matrix \( L = D - A \), where \( D \) is the diagonal matrix whose entries are the degree of the vertices and \( A \) is the adjacency matrix of \( G \). Chung’s \textit{normalized Laplacian} \[2\], \( \mathcal{L} = D^{-1/2}LD^{-1/2} \), is

\[
\mathcal{L}(x, y) = \begin{cases} 
1 & \text{if } x = y, \\
-1/\sqrt{d_x d_y} & \text{if } x \sim y, \\
0 & \text{otherwise}, 
\end{cases}
\]

where \( x \sim y \) denote that \( x \) and \( y \) are adjacent vertices.

If \( \Gamma \) is connected, we know that the eigenvalues of \( \mathcal{L} \) are enumerated by \( 0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \) with the corresponding orthonormal basis of eigenvectors \( \phi_1, \phi_2, \ldots, \phi_n \), as \( n \times 1 \) column vectors. Obviously \( \phi_1(x) = \sqrt{d_x/\text{vol}(\Gamma)} \).

The \textit{normalized Green’s function} \( \mathcal{G} \) was defined in a 2002 paper of Chung and Yau \[3\],

\[
\mathcal{G} = \sum_{\lambda_j > 0} \frac{1}{\lambda_j} \phi_j \phi_j^*. 
\]
This definition of $G$ is equivalent to the two relations
\begin{align}
G\mathcal{L} &= L G = I - P_0, \\
G P_0 &= 0,
\end{align}
where $P_0 = \phi_0 \phi_0^t$ is an $n \times n$ matrix.

In view of the connections to random walks, it is more convenient to use the slightly different Green's function $G$, 
\begin{equation}
G(x, y) = \frac{G(x, y)}{\sqrt{d_x d_y}}.
\end{equation}

Note that both $G$ and $G$ are symmetric matrices.

The hitting time $Q(x, y)$ is the expected number of steps to reach vertex $y$, when started from vertex $x$. Chung and Yau [3] proved the following expression of $Q(x, y)$ in terms of the discrete Green's function and vice versa.

**Theorem 1.1** (Chung–Yau). On a connected graph $\Gamma$, the hitting time $Q(x, y)$ and Green's function $G(x, y)$ satisfy
\begin{align}
Q(x, y) &= \text{vol}(\Gamma) \left( G(y, y) - G(x, y) \right), \\
G(x, y) &= -\frac{1}{\text{vol}(\Gamma)} Q(x, y) + \frac{1}{\text{vol}(\Gamma)^2} \sum_{z \in V(\Gamma)} d_z Q(z, y).
\end{align}

As an illustration of the power of the above theorem, we note that (5) implies a nontrivial symmetry property of hitting times,
\begin{equation}
Q(x, y) + Q(y, z) + Q(z, x) = Q(x, z) + Q(z, y) + Q(y, x)
\end{equation}
for any three vertices $x, y, z$. Eq. (7) was first proved in [11] (see also [5]) as a consequence of the reversibility of the Markov chain for random walks.

The Green's function (with or without boundary) has also been applied to chip firing, load balancing and PageRank (cf. [4,6]).

The paper is organized as following: In Section 2, we introduce two invariants for vertex-weighted graphs and study their basic properties. We also prove an explicit formula for Green's function. In Section 3, we find closed formulas of these two invariants and derive Green's function for some special graphs in a uniform manner. In Section 4, we apply our formula to recover some identities of the hitting time of random walks.

The main results of this paper are Theorem 2.9 and Theorem 4.1.

2. Two invariants of vertex-weighted graphs

We need the following version of Kirchhoff's Matrix-Tree Theorem (cf. [2]).

**Theorem 2.1.** For a graph $\Gamma = (V, E)$ whose eigenvalues of $\mathcal{L}$ are given by $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$, we have
\begin{equation}
\prod_{k=2}^{n} \lambda_k = \frac{\text{vol}(\Gamma) \tau(\Gamma)}{\prod_{v \in V} d_v},
\end{equation}
where $\tau(\Gamma)$ is the number of spanning trees of $\Gamma$.

The following lemma is just a restatement of the fact that the determinant is an alternating summation over permutations of $n$ elements.
Lemma 2.2. Given an \( n \times n \) matrix \( M = (a_{ij}) \), we can define a digraph \( G \) of \( n \) vertices, having an arc from \( i \) to \( j \) of weight \( a_{ij} \) if and only if \( a_{ij} \neq 0 \). A spanning linear subgraph \( \Delta \) of \( G \) has the property that each vertex has exactly one outgoing arc and one incoming arc. Let \( k \) be the number of connected components of \( \Delta \), the weight of \( \Delta \) is defined to be \((-1)^{n+k}\) times the product of the weights of all its arcs. Then \( \det M \) is equal to the sum of the weights of all spanning linear subgraphs of \( G \).

Corollary 2.3. We use the same notation in the above lemma. Let \( 1 \leq i \neq j \leq n \). Let \( M_{ij} \) be the matrix obtained by deleting the \( i \)-th column and the \( j \)-th row of \( M \). Then the \( \Delta \) minor of \( \det M_{ij} \) is equal to the sum of the weights of all spanning subgraphs of \( G \) whose components consist of a directed path from \( i \) to \( j \) and some linear subgraphs.

Proof. If we take \( i = 1 \) and \( j = 2 \), then the assertion follows directly from Lemma 2.2. The general case follows by a relabeling of vertices. \( \square \)

Since the Green’s function is symmetric, we will adapt Lemma 2.2 and Corollary 2.3 to the setting of undirected graphs.

For an undirected simple graph \( \Gamma \), we may associate a directed graph \( \hat{\Gamma} \) with the same vertex set as \( \Gamma \) and an arc \( uv \in E(\hat{\Gamma}) \) if and only if \( u = v \) or \( u \sim v \) in \( \Gamma \). Denote by \( \mathcal{D}(\hat{\Gamma}) \) the totality of spanning subgraphs \( \Delta \) of \( \hat{\Gamma} \) such that each component of \( \Delta \) is either a vertex, an edge or a cycle. \( \mathcal{D}(\hat{\Gamma}) \) is the undirected counterpart of spanning linear subgraphs of \( \hat{\Gamma} \). The only ambiguity is that an undirected \( k \)-cycle \( (k \geq 3) \) has two orientations.

Let \( \Delta \in \mathcal{D}(\Gamma) \) and \((x, y) \in E(\Delta)\). Then we define

\[
\alpha_k(\Delta) = \text{the number of components of } \Delta \text{ with no less than } k \text{ vertices},
\]

\[
\mathcal{I}(\Delta) = \{\text{single-vertex components of } \Delta\},
\]

\[
e(x, y) = \begin{cases} 
2 & \text{if } (x, y) \text{ is a component of } \Delta, \\
1 & \text{otherwise}.
\end{cases}
\]

A vertex-weighted graph is a graph \( \Gamma \) in which each vertex is assigned a weight \( w : V(\Gamma) \to \mathbb{R} \). In this paper, the weight \( w_x \) at \( x \in V(\Gamma) \) will usually be the degree of \( x \) in some ambient graph of \( \Gamma \). So we may assume \( d_x = w_x \), \( \forall x \in V(\Gamma) \), we will denote this weight function \( w \) by \( d_\Gamma \).

Now we introduce two invariants for a vertex-weighted graph \((\Gamma, w)\). For empty graph \( \emptyset \), we define \( R(\emptyset; w) = 1 \) and \( Z(\emptyset; w) = 0 \). When \( \Gamma \neq \emptyset \),

\[
R(\Gamma; w) = \sum_{\Delta \in \mathcal{D}(\Gamma)} (-1)^{\alpha_2(\Delta)} 2^{\alpha_3(\Delta)} \prod_{v \in \mathcal{I}(\Delta)} w_v,
\]

\[
Z(\Gamma; w) = \sum_{\Delta \in \mathcal{D}(\Gamma)} (-1)^{\alpha_2(\Delta)} 2^{\alpha_3(\Delta)} \prod_{v \in \mathcal{I}(\Delta)} w_v \left( \sum_{v \in \mathcal{I}(\Delta)} w_v - \sum_{(x, y) \in E(\Delta)} e(x, y) w_x w_y \right)
+ \sum_{x, y \in V(\Gamma) \atop P \in \mathcal{D}_P(x, y)} w_x w_y R(\Gamma - P; w),
\]

where \( \mathcal{D}_P(x, y) \) is the set of all simple paths (with no repeated vertices) connecting \( x \) and \( y \) in \( \Gamma \). We assume that \( \mathcal{D}_P(x, x) \) consists of the trivial path \( \{x\} \) only. Here \( \Gamma - P \) means removing the vertices of \( P \) together with incident edges.

We will prove in Corollary 2.10 that \( R(\Gamma; d_\Gamma) = 0 \) and \( Z(\Gamma; d_\Gamma) = \text{vol}(\Gamma)^2 \tau(\Gamma) \). Below are two more examples. Let \( \Gamma \) be the single vertex \( pt \) with weight \( a \). Then

\[
R(pt; a) = a, \quad Z(pt; a) = a^2.
\]

Let \( P_2 \) be the two-vertex path with weight \([a, b] \). Then

\[
R(P_2; [a, b]) = ab - 1, \quad Z(P_2; [a, b]) = ab(a + b + 2).
\]
Lemma 2.6. Fix a vertex $x$ effective recursive formulas for computing $R(\Gamma ; w)$ and $Z(\Gamma ; w)$ is a weighted counting of all spanning subgraphs of $\Gamma$ that can be obtained by removing an edge (keeping the end vertices) from some spanning linear subgraph of $\Gamma$. More precisely, if we denote by $B_s$ the following matrix

$$B_s(x, y) = \begin{cases} w_x^2 s + w_x & \text{if } x = y, \\ w_x w_y s - 1 & \text{if } x \sim y, \\ w_x w_y s & \text{otherwise}, \end{cases}$$

then we have $\det B_s = R(\Gamma ; w) + Z(\Gamma ; w) \cdot s$.

The proof of the following lemma is easy.

**Remark 2.4.** In essence, $R(\Gamma ; w)$ and $Z(\Gamma ; w)$ are weighted counting of all spanning subgraphs of $\Gamma$ that can be obtained by removing an edge (keeping the end vertices) from some spanning linear subgraph of $\Gamma$. More precisely, if we denote by $\tilde{B}_s$ the following matrix

$$\tilde{B}_s(x, y) = \begin{cases} w_x^2 s + w_x & \text{if } x = y, \\ w_x w_y s - 1 & \text{if } x \sim y, \\ w_x w_y s & \text{otherwise}, \end{cases}$$

then we have $\det \tilde{B}_s = R(\Gamma ; w) + Z(\Gamma ; w) \cdot s$.

The proof of the following lemma is easy.

**Lemma 2.5.** If $\Gamma$ has $k$ connected components $\Gamma_1, \ldots, \Gamma_k$, then

$$R(\Gamma ; w) = \prod_{i=1}^k R(\Gamma_i ; w), \quad Z(\Gamma ; w) = \sum_{i=1}^k Z(\Gamma_i ; w) \prod_{j=1}^k R(\Gamma_j ; w).$$

The following three lemmas show that these invariants enjoy nice recursive structures and the invariant $Z$ is determined by the invariant $R$. These formulas are easy to check in view of Remark 2.4.

**Lemma 2.6.** Fix a vertex $x \in V(\Gamma)$ and denote by $\mathcal{C}(\Gamma)$ the set of all cycles in $\Gamma$, we have the following effective recursive formulas for computing $R(\Gamma ; w)$ and $Z(\Gamma ; w)$:

$$R(\Gamma ; w) = w_x R(\Gamma - x ; w) - \sum_{y \in V(\Gamma) \setminus x} R(\Gamma - \{x, y\} ; w) - 2 \sum_{C \in \mathcal{C}(\Gamma)} R(\Gamma - C ; w),$$

$$Z(\Gamma ; w) = w_x Z(\Gamma - x ; w) - \sum_{y \in V(\Gamma) \setminus x} Z(\Gamma - \{x, y\} ; w) - 2 \sum_{C \in \mathcal{C}(\Gamma)} Z(\Gamma - C ; w) + w_x^2 R(\Gamma - x ; w) + \sum_{u, v \in V(\Gamma) \setminus x; u \neq v} \sum_{P_1 \in \mathcal{P}(x, u), P_2 \in \mathcal{P}(x, v)} w_u w_v R(\Gamma - \{P_1, P_2\} ; w).$$

**Lemma 2.7.** Fix a vertex $x \in V(\Gamma)$, we have

$$R(\Gamma ; w) = w_x R(\Gamma - x ; w) - \sum_{y \in V(\Gamma) \setminus x} \sum_{P \in \mathcal{P}(x, y)} R(\Gamma - P ; w),$$

$$Z(\Gamma ; w) = w_x Z(\Gamma - x ; w) - \sum_{y \in V(\Gamma) \setminus x} \sum_{P \in \mathcal{P}(x, y)} Z(\Gamma - P ; w) + w_x^2 R(\Gamma - x ; w) + \sum_{u, v \in V(\Gamma) \setminus x; u \neq v} \sum_{P_1 \in \mathcal{P}(x, u), P_2 \in \mathcal{P}(x, v)} w_u w_v R(\Gamma - \{P_1, P_2\} ; w).$$
Lemma 2.8. We have
\[
Z(\Gamma; w) = \sum_{x,y \in V(\Gamma)} \sum_{P \in \mathcal{P}(\Gamma)} w_x w_y R(\Gamma - P; w). \quad (16)
\]

Explicit formulas of \(R(\Gamma; w)\) for certain special graphs with arbitrary weights can be found in Appendix A.

Now we come to prove an explicit formula for the Green's function. The Green's function is the pseudoinverse of the Laplace operator and is singular, so it is not easy to compute directly.

Theorem 2.9. Given a connected graph \(\Gamma\) and \(x, y \in V(\Gamma)\), the Green's function of \(\Gamma\) satisfies
\[
\mathcal{G}(x, y) = \frac{1}{\text{vol}(\Gamma)^2 \tau(\Gamma)} \left( \sum_{P \in \mathcal{P}(\Gamma)} (R(\Gamma - P; d_P) + Z(\Gamma - P; d_P)) - \sum_{u, v \in V(\Gamma)} \sum_{P \in \mathcal{P}(\Gamma)} d_u d_v R(\Gamma - \{P_1, P_2\}; d_P) \right) - \frac{1}{\text{vol}(\Gamma)^2}. \quad (17)
\]

In particular, when \(x = y\),
\[
\mathcal{G}(x, x) = \frac{1}{\text{vol}(\Gamma)^2 \tau(\Gamma)} (R(\Gamma - \{x\}; d_P) + Z(\Gamma - \{x\}; d_P)) - \frac{1}{\text{vol}(\Gamma)^2}. \quad (18)
\]

Proof. By definition,
\[
\mathcal{G}(x, y) = \sum_{i=2}^{n} \frac{1}{\lambda_i} \phi_i(x) \phi_i(y), \quad \forall x, y \in V(\Gamma),
\]
which is equivalent to \(\mathcal{G} = \Phi \text{diag}[0, 1/\lambda_2, \ldots, 1/\lambda_n] \Phi^t\), where \(\Phi = (\phi_1, \ldots, \phi_n)\) is an \(n \times n\) orthogonal matrix. Since \(\phi_1 = (\sqrt{d_1/\text{vol}(\Gamma)}, \ldots, \sqrt{d_n/\text{vol}(\Gamma)})^t\), we have
\[
D^{1/2} J D^{1/2} = \Phi \text{diag}[\text{vol}(\Gamma), 0, \ldots, 0] \Phi^t,
\]
where \(J\) is the \(n \times n\) matrix with all entries equal to 1. From \(\Phi^t L \Phi = \text{diag}[0, \lambda_2, \ldots, \lambda_n]\), we get
\[
\mathcal{G} = (L + D^{1/2} J D^{1/2})^{-1} - \frac{1}{\text{vol}(\Gamma)^2} D^{1/2} J D^{1/2}.
\]
In terms of \(\mathcal{G}\), we have
\[
\mathcal{G} = D^{-1/2} \mathcal{G} D^{-1/2} = D^{-1} M^{-1} D^{-1} - \frac{1}{\text{vol}(\Gamma)^2} J, \quad (19)
\]
where \(M = M_1 = D^{-1/2} (L + D^{1/2} J D^{1/2}) D^{-1/2}\) is given by
\[
M_s(x, y) = \begin{cases} 
  s + 1/d_x & \text{if } x = y, \\
  s - 1/d_x d_y & \text{if } x \sim y, \\
  s & \text{otherwise.}
\end{cases} \quad (20)
\]
By definition,
\[
\det M_s \prod_{x \in V(\Gamma)} d_x^2 = R(\Gamma; d_P) + Z(\Gamma; d_P) \cdot s. \quad (21)
\]
Moreover, by the Matrix-Tree Theorem, we have
\[
\det M = \frac{\text{vol}(\Gamma) \prod_{i=2}^{n} \lambda_i}{\prod_{x \in V(\Gamma)} d_x} = \frac{\text{vol}(\Gamma)^2 \tau(\Gamma)}{\prod_{x \in V(\Gamma)} d_x^2}.
\]
Since the entries of \((M^{-1})\) are given by \((-1)^{i+j} \det M_{ij} / \det M\), from the graph-theoretic explanation of the \((i, j)\) minor \(\det M_{ij}\) in Corollary 2.3, we see that (19) implies (17).

**Corollary 2.10.** We have \(R(\Gamma; d_{\Gamma}) = 0\) and if \(\Gamma\) is connected, we have
\[
Z(\Gamma; d_{\Gamma}) = \text{vol}(\Gamma)^2 \tau(\Gamma).
\]  

**Proof.** For the \(M_s\) defined in (20), we have
\[
M_s = D^{-1/2}(L + D^{1/2} \mathcal{J} D^{1/2}) D^{-1/2} = D^{-1/2} \Phi \text{ diag}[\text{vol}(\Gamma)s, \lambda_2, \ldots, \lambda_n] \Phi^t D^{-1/2},
\]
so its determinant equals
\[
\det M_s = \frac{\text{vol}(\Gamma)^2 \tau(\Gamma) \cdot s}{\prod_{x \in V(\Gamma)} d_x^2}.
\]  
Then the corollary follows from (21).

Recall the following well-known results from linear algebra.

**Lemma 2.11.** Let \(M\) be an \(n \times n\) matrix whose rows and columns all sum to zero. Then all cofactors \((-1)^{i+j} \det(M_{ij})\) of \(M\) are equal.

**Lemma 2.12.** Let \(\Gamma\) be a connected graph with Laplacian matrix \(L\). Let \(L'\) denote the matrix obtained by deleting the first row and column from \(L\). Then:
\[
\tau(\Gamma) = \det(L').
\]

We now prove one more identity about the invariant \(R\).

**Lemma 2.13.** Let \(\Gamma\) be a connected graph. For any \(x, y \in V(\Gamma)\), we have
\[
R(\Gamma - \{x\}; d_{\Gamma}) = \sum_{P \in \mathcal{P}_\Gamma(x, y)} R(\Gamma - P; d_{\Gamma}) = \tau(\Gamma).
\]  

**Proof.** Recall that the Laplacian \(L\) of \(\Gamma\) is given by
\[
L(x, y) = \begin{cases} 
d_x & \text{if } x = y, \\
-1 & \text{if } x \sim y, \\
0 & \text{otherwise.}
\end{cases}
\]

We may assume that \(x\) and \(y\) correspond to the first and second row-indices of \(L\) respectively. Then \(L_{xy}\) is the matrix obtained by deleting the first column and second row of \(L\). Then since \(L\) has the property that each row or each column adds up to zero, by Lemma 2.11 and Lemma 2.12, we have \(\det(L_{xx}) = \det(L_{yy}) = -\det(L_{xy}) = \tau(\Gamma)\).

On the other hand, by Lemma 2.2 and Corollary 2.3, we have \(R(\Gamma - \{x\}; d_{\Gamma}) = \det(L_{xx})\) and \(\sum_{P \in \mathcal{P}_\Gamma(x, y)} R(\Gamma - P; d_{\Gamma}) = -\det(L_{xy})\). So (24) is proved.
Remark 2.14. As pointed out by a referee, Lemma 2.13 and Lemma 2.2 imply the following: Fix a vertex $x$ of $\Gamma$. Let $H$ be the digraph that has the same vertex set as $\Gamma$ and $uv \in E(H)$ if and only if $u \sim v$ in $\Gamma$ (note that $H$ has no loops). Then

$$\tau(\Gamma) = \sum_{x \notin \Delta} (-1)^{p(\Delta)} \prod_{v \in \mathcal{V}(\Gamma), v \notin \Delta, v \neq x} d_v,$$

where $\Delta$ runs over (not necessarily spanning) linear subgraphs (including $\emptyset$) of $H$ avoiding $x$, $p(\Delta)$ is the number of components of $\Delta$ and $d_v$ is the degree of $v$ in $\Gamma$, which equals the indegree (or outdegree) of $v$ in $H$.

3. Green’s functions of some special graphs

The weight function of a labeled vertex-weighted graph (with vertices labeled $1, \ldots, n$) can be identified with a sequence $w = [w_1, \ldots, w_n]$. For brevity, in the proofs of the following lemmas, we use $r_n$ and $z_n$ to denote $R(\Gamma_n; w)$ and $Z(\Gamma_n; w)$ respectively. The proofs are the same, we use the recursive formulas (12), (13) and the initial values (9) to get the explicit formulas.

Lemma 3.1. For a path on $n$ vertices $P_n$ with weight $w = [2, \ldots, 2] = [2^n]$,

$$R(P_n; [2^n]) = n + 1,$$

$$Z(P_n; [2^n]) = \frac{1}{3} n(n + 1)^2(n + 2).$$

Proof. Let $x$ be the leftmost vertex of $P_n$. By (12), we have

$$r_n = 2r_{n-1} - r_{n-2},$$

which implies $r_n = n + 1$. By (13), we have

$$z_n = 2z_{n-1} - z_{n-2} + 4n + 8 \sum_{i=0}^{n-2} (i + 1)$$

$$= 2z_{n-1} - z_{n-2} + 4n^2.$$

It is not difficult to solve the above recursion formula to get $z_n$. \hfill \Box

Lemma 3.2. Let $w = [1, 2, \ldots, 2] = [1, 2^{n-1}]$. Then

$$R(P_n; [1, 2^{n-1}]) = 1,$$

$$Z(P_n; [1, 2^{n-1}]) = \frac{4}{3} n^3 - \frac{1}{3} n.$$

Here the leftmost vertex of $P_n$ is assigned weight 1.

Proof. Let $x$ be the leftmost vertex of $P_n$. By (12) and (25), we directly get $r_n = 1$. Similarly, by (13) and (26), we directly get $z_n$. \hfill \Box

Lemma 3.3. Let $K_n, n \geq 1$ be the complete graph on $n$ vertices. Then for any $m$, we have

$$R(K_n; m^n) = (m - n + 1)(m + 1)^{n-1},$$

$$Z(K_n; m^n) = n \cdot m^2 (m + 1)^{n-1}.$$
Proof. Let $x$ be an arbitrary vertex of $K_n$. By (12), we have

$$r_n = mr_{n-1} - (n-1)r_{n-2} - \sum_{i=2}^{n-1} \binom{n-1}{i} i!r_{n-1-i},$$

which can be solved to get the desired $r_n$ in (27). By (13), we have

$$z_n = mz_{n-1} - (n-1)z_{n-2} - \sum_{i=2}^{n-1} \binom{n-1}{i} i!z_{n-1-i} + m^2r_{n-1}$$

$$+ m^2 \sum_{i=1}^{n-1} \binom{n-1}{i} (i+1)!r_{n-1-i}.$$

We can check that the above equation is satisfied by the right-hand side of (28).

Lemma 3.4. Let $S_n$, $n \geq 1$ be the $n$-star graph with weight vector $[m, 1^{n-1}]$ that assigns $m$ to the center vertex and 1 to $n-1$ leaves. Then

$$R(S_n; [m, 1^{n-1}]) = m - n + 1,$$

$$Z(S_n; [m, 1^{n-1}]) = m^2 - 3m + 3mn.$$

Proof. We take $x$ to be the center vertex of $S_n$. Then by (10) and (11), we directly get $r_n = m - n + 1$ and

$$z_n = m(n-1) - (n-1)(n-2) + m^2 + 2\left(m(n-1) + \binom{n-1}{2}\right)$$

$$= m^2 - 3m + 3mn.$$

One can also take $x$ to be a leaf, then we will get recursive formulas instead, not so direct as above.

For brevity, in the following examples, we will denote by $A$ and $B$ the two sums on the right-hand side of (17) respectively,

$$A := \sum_{P \in \mathcal{P}(x,y)} \left(R(\Gamma - P; d_{\Gamma}) + Z(\Gamma - P; d_{\Gamma})\right),$$

$$B := \sum_{u, v \in V(\Gamma)} \sum_{P_1 \in \mathcal{P}(x,u), P_2 \in \mathcal{P}(y,v)} d_u d_v R(\Gamma - \{P_1, P_2\}; d_{\Gamma}).$$

We will show that Theorem 2.9 is effective in obtaining closed formulas of Green’s functions, hence the hitting time of random walk via Theorem 1.1. We need the explicit formulas of $R(\Gamma; w)$ and $Z(\Gamma; w)$ derived in the previous lemmas.

Example 3.5. The Green’s function of $C_n$, the cycle on $n \geq 3$ vertices, has been computed by Ellis [6]. Here we recover it by applying Theorem 2.9 and Lemma 3.1. Let $P_n$ denote the path on $n \geq 1$ vertices. First we have

$$G(x, x) = \frac{1}{(2n)^2} \cdot n \left(R(P_{n-1}; 2^{n-1}) + Z(P_{n-1}; 2^{n-1})\right) - \frac{1}{(2n)^2}$$

$$= \frac{(n + 1)(n - 1)}{12n}.$$
Since $G(x, y)$ depends only on $|x - y|$, we may assume $x = 0$, $y = j$. For the first sum on the right-hand side of (17), we have
\begin{align*}
A & := R(P_{n-j-1}; 2^{n-j-1}) + R(P_{j-1}; 2^{j-1}) + Z(P_{n-j-1}; 2^{n-j-1}) + Z(P_{j-1}; 2^{j-1}) \\
& = \frac{1}{3}(j^4 + (n-j)^4) - \frac{1}{3}(j^2 + (n-j)^2) + n.
\end{align*}

For the second sum on the right-hand side of (17), we have
\begin{align*}
B & := 4 \left( j(n-j) + (n-j) \sum_{i=1}^{j-1} i(j-i+1) + j \sum_{i=1}^{n-j-1} i(n-j-i+1) \\
& \quad + \sum_{k=j+1}^{n-1} \sum_{\ell=1}^{j-1} ((k-j)\ell + (n-k)(j-\ell)) \right) \\
& = \frac{2}{3} j(j-n)(j^2-nj-1-n^2).
\end{align*}

We get the Green's function of $C_n$,
\begin{align*}
G(x, y) &= \frac{1}{(2n)^2} (A - B) - \frac{1}{(2n)^2} \\
& = \frac{n+1(n-1)}{12n} - \frac{j(n-j)}{2n} \\
& = \frac{n+1(n-1)}{12n} - \frac{|x-y|(n-|x-y|)}{2n}.
\end{align*}

**Example 3.6.** The Green's function of $P_n$ (with more general weights), has been computed by Chung and Yau [3]. By Lemma 3.2, for any $1 \leq x \leq y \leq n$,
\begin{align*}
A & = 1 + \frac{4}{3} (n-y)^3 - \frac{1}{3} (n-y) + \frac{4}{3} (x-1)^3 - \frac{1}{3} (x-1), \\
B & = \frac{4}{3} (x^3 - y^3) - (2n+2)(x^2 - y^2) + \left( 4n - \frac{1}{3} \right) (x-y).
\end{align*}

We get the Green's function of $P_n$,
\begin{align*}
G(x, y) &= \frac{1}{12(n-1)} \left( 6(x-1)^2 + 6(n-y)^2 - 2n^2 + 4n - 3 \right).
\end{align*}

**Example 3.7.** For the complete graph $K_n$ on $n$ vertices, we have $\text{vol}(K_n) = n(n-1)$ and $\tau(K_n) = n^{n-2}$. We denote by $r_{n,m}$ and $z_{n,m}$ the invariants $R(K_n; m^5)$ and $Z(K_n; m^5)$ respectively. When $x \neq y \in V(K_n)$, we have
\begin{align*}
G(x, y) &= \frac{1}{\text{vol}(K_n)^2 \tau(K_n)} \left( \sum_{i=0}^{n-2} \binom{n-2}{i} i!(r_{n-2-i,n-1} + z_{n-2-i,n-1}) \\
& \quad + (n-1)^2 \sum_{i=0}^{n-2} \binom{n-2}{i} (i+1)! r_{n-2-i,n-1} \right) - \frac{1}{\text{vol}(K_n)^2} \\
& = \frac{1}{n^2(n-1)^2} \left( -n^n + 2n^{n-1} \right) - \frac{1}{n^2(n-1)^2} \\
& = -\frac{1}{n^2}.
\end{align*}
Similarly, we have

\[ G(x, x) = \frac{1}{\text{vol}(K_n)^2} \tau(K_n) \left( r_{n-1, n-1} - z_{n-1, n-1} \right) - \frac{1}{\text{vol}(K_n)^2} \frac{n}{n^2}. \]

**Example 3.8.** Let \( c \) be the center of the star \( S_n \) and \( x, y \) distinct leaves of \( S_n \). Then from Theorem 2.9 and Lemma 3.4, it is not difficult to get

\[ G(c, c) = \frac{1}{4(n-1)}, \quad G(x, x) = \frac{4n-7}{4(n-1)}, \]
\[ G(c, x) = \frac{-1}{4(n-1)}, \quad G(x, y) = \frac{-3}{4(n-1)}. \]

By (5), we get the hitting times of random walks on \( S_n \).

\[ Q(c, x) = 2n - 3, \quad Q(x, c) = 1, \quad Q(x, y) = 2n - 2. \]

**4. The hitting time of random walks**

By Theorem 1.1, Theorem 2.9 and Lemma 2.13, we immediately get the following explicit formula for the hitting time of random walks.

**Theorem 4.1.** Given a connected graph \( \Gamma \) and \( x, y \in V(\Gamma) \), the expected hitting time \( Q(x, y) \) satisfies

\[
Q(x, y) = \frac{1}{\text{vol}(\Gamma) \tau(\Gamma)} \left( Z(\Gamma - \{y\}; d_\Gamma) - \sum_{P \in \mathcal{P}_\Gamma(x, y)} Z(\Gamma - P; d_\Gamma) \right) \\
+ \sum_{u, v \in V(\Gamma)} \sum_{\substack{P_1 \in \mathcal{P}_\Gamma(x, u) \cap \mathcal{P}_\Gamma(y, v) \cap P_1 \cap P_2 = \emptyset \atop u \neq v}} d_u d_v R(\Gamma - P_1, P_2; d_\Gamma). \tag{29}
\]

**Corollary 4.2.** Under the above notation, we have

\[
Q(x, y) - Q(y, x) = \frac{1}{\text{vol}(\Gamma) \tau(\Gamma)} \left( Z(\Gamma - \{y\}; d_\Gamma) - Z(\Gamma - \{x\}; d_\Gamma) \right).
\]

**Corollary 4.3.** Given a connected graph \( \Gamma \) and \( x, y \in V(\Gamma) \), we have

\[
Q(x, y) = \frac{1}{\text{vol}(\Gamma) \tau(\Gamma)} \sum_{u, v \in V(\Gamma)} d_u d_v \left( \sum_{P \in \mathcal{P}_\Gamma(u, v)} \tau(\Gamma\{P, y\}) \right) \\
- \sum_{\substack{P_1 \in \mathcal{P}_\Gamma(x, y) \cap \mathcal{P}_\Gamma(u, v) \cap P_1 \cap P_2 = \emptyset \atop u \neq v}} \tau(\Gamma\{P_1, P_2\}) + \sum_{\substack{P_1 \in \mathcal{P}_\Gamma(x, y) \cap \mathcal{P}_\Gamma(u, v) \cap P_1 \cap P_2 = \emptyset \atop u \neq v}} \tau(\Gamma\{P_1, P_2\}) \tag{30}
\]

where \( \Gamma\{P, y\} \) denotes the graph obtained by contracting \( P \) and \( y \) to a point and similarly for \( \Gamma\{P_1, P_2\} \). Note that \( \Gamma\{P, y\} \) and \( \Gamma\{P_1, P_2\} \) may be multigraphs.

**Proof.** Note that Lemma 2.12 holds also for multigraphs without loops. Then (30) follows readily from (29) and Lemma 2.8. \( \square \)
Remark 4.4. Let $e$ be an edge of a connected multigraph $\Gamma$ (without loops). It is well known that $\tau(\Gamma) = \tau(\Gamma - e) + \tau(\Gamma/e)$, which can be combined with (30) to give an algorithm for computing $Q(x,y)$.

A powerful approach to random walks is from the viewpoint of electric networks. Tetali [9] obtained a remarkable formula of the hitting time in terms of the effective resistance of electrical networks. See also [10] for an alternative proof and other interesting results. Tetali's formula was applied by Chen and Zhang [1] and Palacios [8] to obtain explicit closed formulas for hitting times on trees or unicycle graphs. For general graphs, there seems no such explicit formulas known to the authors.

We expect that Theorem 4.1 and the properties of $R(\Gamma, w), Z(\Gamma, w)$ proved in Section 2 will lead to more intuitive proofs of most existing results on hitting times. As an example, we give a proof the following well-known result on the expected return time, i.e., the expected number of steps for a walk to return to the starting vertex after leaving it.

**Theorem 4.5.** Let $\Gamma = (V,E)$ be a connected graph. Then the expected return time to a vertex $x \in V(\Gamma)$ is equal to

$$1 + \frac{1}{d_x} \sum_{y \sim x} Q(y,x) = \frac{\text{vol}(\Gamma)}{d_x}. \quad (31)$$

**Proof.** By Theorem 4.1, Eq. (15), Lemma 2.13 and Corollary 2.10, we have

$$\sum_{y \sim x} Q(y,x) = \frac{1}{\text{vol}(\Gamma) \tau(\Gamma)} \left( d_x Z(\Gamma - \{x\}; d_\Gamma) - \sum_{y \in V(\Gamma) \atop y \sim x} \sum_{P \in \mathcal{P}_R(x,y)} Z(\Gamma - P; d_\Gamma) \right) + \sum_{y \sim x} \sum_{u \in V(\Gamma) \atop u \neq y} \sum_{P_1 \in \mathcal{P}_R(x,u) \atop P_1 \cap P_2 = \emptyset} \sum_{P_2 \in \mathcal{P}_R(y,v) \atop P_1 \cap P_2 = \emptyset} d_x d_v R(\Gamma - \{P_1, P_2\}; d_\Gamma)$$

$$= \frac{1}{\text{vol}(\Gamma) \tau(\Gamma)} \left( Z(\Gamma; d_\Gamma) - d_x^2 R(\Gamma - \{x\}; d_\Gamma) - \sum_{v \neq x} \sum_{P \in \mathcal{P}_R(x,v)} d_x d_v R(\Gamma - P; d_\Gamma) \right)$$

$$= \frac{1}{\text{vol}(\Gamma) \tau(\Gamma)} \left( Z(\Gamma; d_\Gamma) - \sum_{v \in V(\Gamma) \atop v \neq x} d_x d_v R(\Gamma - P; d_\Gamma) \right)$$

$$= \text{vol}(\Gamma) - \frac{1}{\text{vol}(\Gamma) \tau(\Gamma)} d_x \sum_{v \in V(\Gamma)} d_v R(\Gamma - \{x\}; d_\Gamma)$$

$$= \text{vol}(\Gamma) - d_x,$$

which gives (31). □

We also have the following well-known result about the mean value of hitting times between adjacent vertices.
Corollary 4.6. Let \( \Gamma = (V, E) \) be a connected graph with \( n \) vertices. Then
\[
\sum_{x \in V(\Gamma)} \sum_{y \in V(\Gamma)} Q(x, y) = (n - 1) \text{vol}(\Gamma). \tag{32}
\]

Proof. By symmetry of \( x, y \) and (31), we have
\[
\sum_{x \in V(\Gamma)} \sum_{y \in V(\Gamma)} Q(x, y) = \sum_{x \in V(\Gamma)} \sum_{y \sim x(\Gamma)} Q(y, x) = n \cdot \text{vol}(\Gamma) - \sum_{x \in V(\Gamma)} dx
\]
\[
= (n - 1) \text{vol}(\Gamma),
\]
as claimed. \( \square \)

Now let us consider random walks on a tree \( T \). For any two vertices \( a, b \in V(T) \), there is a unique path between \( a \) and \( b \), whose length is the distance \( d(a, b) \) between \( a \) and \( b \).

Lemma 4.7. Given a tree \( T \) and a vertex \( a \in V(T) \), define a weight function \( d_{T,a^{+k}} \) on \( V(T) \) by
\[
d_{T,a^{+k}}(x) = \begin{cases} 
  d_a + k, & \text{if } x = a, \\
  d_x, & \text{if } x \neq a.
\end{cases}
\]
Then \( R(T; d_{T,a^{+k}}) = k \).

Proof. We prove this by induction on \( |V(T)| \). When \( T \) is a single point, the equation obviously holds. From (12) and by induction, we have
\[
R(T; d_{T,a^{+k}}) = (d_a + k)R(T - a; d_{T,a^{+k}}) - \sum_{x \in V(\Gamma)} R(T - \{x, a\}; d_{T,a^{+k}})
\]
\[
= d_a + k - d_a = k,
\]
as claimed. \( \square \)

Lemma 4.8. Given a tree \( T \) and two vertices \( a, b \in V(T) \), define a weight function \( d_{T,a^{+},b^{+}} \) on \( V(T) \) by
\[
d_{T,a^{+},b^{+}}(x) = \begin{cases} 
  d_a + 1, & \text{if } x = a, \\
  d_b + 1, & \text{if } x = b, \\
  d_x, & \text{if } x \neq a, b.
\end{cases}
\]
Then \( R(T; d_{T,a^{+},b^{+}}) = d(a, b) + 2 \).

Proof. We prove this by induction on \( |V(T)| \). When \( T \) is a two-vertex path, the equation obviously holds. From (12) and Lemma 4.7, we finally get
\[
R(T; d_{T,a^{+},b^{+}}) = (d_a + 1)R(T - a; d_{T,a^{+},b^{+}}) - \sum_{x \in V(\Gamma)} R(T - \{x, a\}; d_{T,a^{+},b^{+}})
\]
\[
= (d_a + 1)(d(a, b) + 1) - (d_a - 1)(d(a, b) + 1) - d(a, b)
\]
\[
= d(a, b) + 2,
\]
as claimed. \( \square \)
From Theorem 4.1, we can prove the following explicit formula for the hitting time on trees, which was first obtained by Chen and Zhang [1] using the method of electric networks.

**Theorem 4.9.** Let $T$ be a tree and $a, b \in V(T)$ with $P_{a \rightarrow b}$ the unique path connecting $a$ to $b$. For any $u \in V(P_{a \rightarrow b})$, we denote by $T_u$ the component of $T - E(P_{a \rightarrow b})$ that contains $u$. Then the hitting time $Q(a, b)$ satisfies

$$Q(a, b) = d(a, b)^2 + 2 \sum_{u \in V(P_{a \rightarrow b})} |E(T_u)|d(u, b).$$  \hspace{1cm} (33)

**Proof.** For a tree $T$, Eq. (29) becomes

$$Q(a, b) = \frac{1}{\text{vol}(T)} \left( Z(T - \{b\}; d_T) - Z(T - P_{a \rightarrow b}; d_T) \right)$$

$$+ \sum_{u, v \in V(T)} d_ud_v R(T - \{P_{u \rightarrow a}, P_{v \rightarrow b}\}; d_T).$$ \hspace{1cm} (34)

By applying Lemma 2.8, it is not difficult to verify that

$$Z(T - \{b\}; d_T) - Z(T - P_{a \rightarrow b}; d_T)$$

$$= \sum_{u, v \in V(P_{a \rightarrow b})} \min(d(u, b), d(v, b)) \left( \sum_{x \in T_u} d_x \right) \left( \sum_{x \in T_v} d_x \right).$$ \hspace{1cm} (35)

On the other hand, we can verify that

$$\sum_{u, v \in V(T)} d_ud_v R(T - \{P_{u \rightarrow a}, P_{v \rightarrow b}\}; d_T)$$

$$= \frac{1}{2} \sum_{u, v \in V(P_{a \rightarrow b})} d(u, v) \left( \sum_{x \in T_u} d_x \right) \left( \sum_{x \in T_v} d_x \right).$$ \hspace{1cm} (36)

Adding up (35) and (36), we get

$$Q(a, b) = \frac{1}{\text{vol}(T)} \left( \sum_{u, v \in V(P_{a \rightarrow b})} d(u, b) \left( \sum_{x \in T_u} d_x \right) \left( \sum_{x \in T_v} d_x \right) \right)$$

$$= \sum_{u \in V(P_{a \rightarrow b})} d(u, b) \sum_{x \in T_u} d_x$$

$$= 2 \sum_{u \in V(P_{a \rightarrow b})} |E(T_u)|d(u, b) + d(a, b) + \sum_{i=1}^{d(a, b)-1} 2i$$

$$= d(a, b)^2 + 2 \sum_{u \in V(P_{a \rightarrow b})} |E(T_u)|d(u, b),$$

as claimed. \hspace{1cm} $\square$

**Remark 4.10.** We may also use Corollary 4.3 to prove (33). We simply need to note that the number of spanning trees of a unicycle graph equals the length of the unique cycle.

Finally, we give a proof of the following identity of commute times, which was previously proved using the “topological formulas” from the electric networks (cf. [7]).
**Theorem 4.11.** Let \( x, y \in V(\Gamma) \) be two distinct vertices and \( \Gamma' \) be the graph obtained from \( \Gamma \) by identifying \( x \) and \( y \). Then we have

\[
Q(x, y) + Q(y, x) = \frac{\tau(\Gamma')}{\tau(\Gamma)}.
\]

**Proof.** First note that the Matrix-Tree Theorem 2.1 also holds for multigraphs (with loops or multi-edges). A loop at \( x \) adds 2 to \( d_x \). We may assume \( x \neq y \). Let \( \Gamma'' \) be the graph obtained from \( \Gamma \) by identifying \( x \) and \( y \). This new vertex will be denoted by \( z \). Of course \( \Gamma'' \) may be a multigraph. Let \( u, v \) be distinct vertices of \( \Gamma'' \) different from \( z \) and denote by \( M'_s \) a symmetric matrix specified by

\[
M'_s(u, u) = d_u^2 s + d_u, \quad M'_s(u, v) = d_u d_v s - m_{uv},
\]

\[
M'_s(z, z) = (d_x + d_y)^2 s + d_x + d_y - 2m_{xy},
\]

\[
M'_s(z, u) = (d_x + d_y) d_u s - m_{ux} - m_{uy},
\]

where \( m_{xy} = 1 \) if \( x \) and \( y \) are adjacent in \( \Gamma \) and \( m_{xy} = 0 \) otherwise.

Let \( D' \) and \( A' \) denote the degree matrix and adjacency matrix of \( \Gamma'' \) respectively. The normalized Laplacian \( \mathcal{L}' \) of \( \Gamma'' \) is given by \( \mathcal{L}' = D'^{-1/2}(D' - A')D'^{-1/2} \). Since \( \Gamma'' \) is connected, we know that \( \mathcal{L}' \) also has a simple eigenvalue 0. Note that \( M'_s = D'^{-1/2}(\mathcal{L}' + D'^{-1/2} J D'^{-1/2})D'^{-1/2} \).

By the same argument used in the proof of Theorem 2.9, we get

\[
\det M'_s = \vol(\Gamma')^2 \tau(\Gamma') \cdot s.
\]

On the other hand, it is not difficult to check that the coefficient of \( s \) in \( \det M'_s \) is equal to

\[
Z(\Gamma - \{x\}; d_{\Gamma}) + Z(\Gamma - \{y\}; d_{\Gamma}) - 2 \sum_{P \in \mathcal{P}_{\Gamma}(x, y)} Z(\Gamma - P; d_{\Gamma}) + 2 \sum_{u, v \in V(\Gamma)} \sum_{\substack{P_1 \in \mathcal{P}_{\Gamma}(x, u) \cap \mathcal{P}_{\Gamma}(y, v) \cap P_1 \cap P_2 = \emptyset \atop P_2 \in \mathcal{P}_{\Gamma}(y, v)}} d_u d_v R(\Gamma - \{P_1, P_2\}; d_{\Gamma})
\]

\[
= (Q(x, y) + Q(y, x)) \vol(\Gamma) \tau(\Gamma).
\]

Finally we get

\[
Q(x, y) + Q(y, x) = \frac{\vol(\Gamma')^2 \tau(\Gamma')}{\vol(\Gamma) \tau(\Gamma)} = \frac{\vol(\Gamma') \tau(\Gamma')}{\tau(\Gamma)}.
\]

where we used \( \vol(\Gamma') = \vol(\Gamma'). \) \( \square \)

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**Appendix A. Explicit formulas of** \( R(\Gamma; w) \)

In this appendix, we present explicit formulas of \( R(\Gamma; w) \) for certain special graphs since they are useful in the recursive computation of the hitting time of random walks. By definition \( R(\Gamma; w) = \det(B) \), where \( B \) is the \( n \times n \) matrix

\[
B(x, y) = \begin{cases} 
  w_x, & \text{if } x = y, \\
  -1, & \text{if } x \sim y, \\
  0, & \text{otherwise}.
\end{cases}
\]
Proposition A.1. We have the following explicit formulas for $R(Γ; w)$:

(i) Let $P_n, n \geq 1$ be the path on $n$ vertices. Then

$$R(P_n; [w_1, \ldots, w_n]) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{1 \leq i_1 < \cdots < i_{n-2k} \leq n \atop i_j-i_{j-1} \equiv 1 \pmod{2}, 2 \leq j \leq n-2k \atop i_{n-2k} \equiv n, i_1 \equiv 1 \pmod{2}} (-1)^k w_{i_1} \cdots w_{i_{n-2k}}.$$ 

In particular, we have

$$R(P_1; [w_1]) = w_1, \quad R(P_2; [w_1, w_2]) = w_1 w_2 - 1, \quad R(P_3; [w_1, w_2, w_3]) = w_1 w_2 w_3 - w_1 - w_3.$$ 

(ii) Let $C_n, n \geq 3$ be the cycle on $n$ vertices. Then

$$R(C_n; [w_1, \ldots, w_n]) = \Phi_n + \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{1 \leq i_1 < \cdots < i_{n-2k} \leq n \atop i_j-i_{j-1} \equiv 1 \pmod{2}, 2 \leq j \leq n-2k \atop i_{n-2k} \equiv n, i_1 \equiv 1 \pmod{2}} (-1)^k w_{i_1} \cdots w_{i_{n-2k}},$$

where $\Phi_n$ is given by

$$\Phi_n = \begin{cases} 
-2 & \text{if } n \equiv 1 \pmod{2}, \\
0 & \text{if } n \equiv 0 \pmod{4}, \\
-4 & \text{if } n \equiv 2 \pmod{4}.
\end{cases}$$

In particular, we have

$$R(C_3; [w_1, w_2, w_3]) = w_1 w_2 w_3 - w_1 - w_2 - w_3 - 2, \quad R(C_4; [w_1, w_2, w_3, w_4]) = w_1 w_2 w_3 w_4 - w_1 w_2 - w_2 w_3 - w_3 w_4 - w_1 w_4.$$ 

(iii) Let $K_n, n \geq 1$ be the complete graph on $n$ vertices. Then

$$R(K_n; [w_1, \ldots, w_n]) = w_1 \cdots w_n - \sum_{k=0}^{n-2} (n-k-1)e_k(w_1, \ldots, w_n),$$

where $e_k$ is the $k$-th elementary symmetric polynomial.

(iv) Let $S_n, n \geq 1$ be the star on $n$ vertices. Then

$$R(S_n; [c, w_1, \ldots, w_{n-1}]) = c w_1 \cdots w_{n-1} - \sum_{i=1}^{n-1} \prod_{j=1, j \neq i}^{n-1} w_j,$$

where $c$ is the weight of the center vertex.

Proof. From (12) or (14), we have

$$R(P_n; [w_1, \ldots, w_n]) = w_n R(P_{n-1}; [w_1, \ldots, w_{n-1}]) - R(P_{n-2}; [w_1, \ldots, w_{n-2}]),$$

which implies (i) by an inductive argument. The rest of the proposition can be proved similarly. We omit the details. \qed
Let $T$ denote a tree. Then Lemma 4.8 may be strengthened as follows:

**Proposition A.2.** Given a tree $T$ and two vertices $a, b \in V(T)$, define a weight function $d_{T,a+k,b+\ell}$ on $V(T)$ by

$$d_{T,a+k,b+\ell}(x) = \begin{cases} d_a + k, & \text{if } x = a, \\ d_b + \ell, & \text{if } x = b, \\ d_x, & \text{if } x \neq a, b. \end{cases}$$

Then $R(T;d_{T,a+k,b+\ell}) = k\ell \cdot d(a,b) + k + \ell$.

**Proof.** Use the proof of Lemma 4.8. \qed

**Proposition A.3.** Given a tree $T$ and three vertices $a, b, c \in V(T)$, denote by $v$ the unique common vertex of the three paths $P_{a\rightarrow b}$, $P_{a\rightarrow c}$, $P_{b\rightarrow c}$. Define a weight function $d_{T,a+k,b+\ell,c+m}$ on $V(T)$ by

$$d_{T,a+k,b+\ell,c+m}(x) = \begin{cases} d_a + k, & \text{if } x = a, \\ d_b + \ell, & \text{if } x = b, \\ d_c + m, & \text{if } x = c, \\ d_x, & \text{if } x \neq a, b, c. \end{cases}$$

Then

$$R(T;d_{T,a+k,b+\ell,c+m}) = k\ell m(d(v,a)d(v,b) + d(v,a)d(v,c) + d(v,b)d(v,c)) + k\ell \cdot d(a,b) + km \cdot d(a,c) + \ell m \cdot d(b,c) + k + \ell + m.$$  

**Proof.** First assume that $v \neq a, b, c$, from (14) and Proposition A.2, we have

$$R(T;d_{T,a+k,b+\ell,c+m}) = d_v R(T-x;a+k,b+\ell,c+m) - \sum_{u \in V(T)} R(T-\{u,v\}; a+k,b+\ell,c+m)$$

$$= d_v (k(d(v,a) - 1) + k + 1)(\ell(d(v,b) - 1) + \ell + 1)(m(d(v,c) - 1) + m + 1)$$

$$- (d_v - 3)(k(d(v,a) - 1) + k + 1)(\ell(d(v,b) - 1) + \ell + 1)(m(d(v,c) - 1) + m + 1)$$

$$- (k(d(v,a) - 2) + k + 1)(\ell(d(v,b) - 1) + \ell + 1)(m(d(v,c) - 1) + m + 1)$$

$$- (k(d(v,a) - 1) + k + 1)(\ell(d(v,b) - 2) + \ell + 1)(m(d(v,c) - 1) + m + 1)$$

$$- (k(d(v,a) - 1) + k + 1)(\ell(d(v,b) - 1) + \ell + 1)(m(d(v,c) - 2) + m + 1)$$

$$= k\ell m(d(v,a)d(v,b) + d(v,a)d(v,c) + d(v,b)d(v,c)) + k\ell \cdot d(a,b) + km \cdot d(a,c) + \ell m \cdot d(b,c) + k + \ell + m.$$  

If $v$ equals one of $a, b, c$, the proof is similar. \qed

Let $\Gamma$ be a connected unicycle graph. The unique cycle is denoted by $C$ and $V(C) = \{1,2,\ldots,p\}$. Let $T_i$ ($i = 1,2,\ldots,p$) be the tree component of $\Gamma\setminus E(C)$ containing $i$. The following two propositions may be used to recover Chen–Zhang’s formula [4] for the hitting time of random walks on unicycle graphs.

**Proposition A.4.** Let $\Gamma$ be a connected unicycle graph. For $a \in V(\Gamma)$, define a weight function $d_{\Gamma,a+k}$ on $V(\Gamma)$ by

$$d_{\Gamma,a+k}(x) = \begin{cases} d_a + k, & \text{if } x = a, \\ d_x, & \text{if } x \neq a. \end{cases}$$

We have $R(\Gamma;d_{\Gamma,a+k}) = kp$.  

Proof. If \( a \in V(C) \), then by (12), Lemma 4.7 and Lemma 4.8, we have
\[
R(\Gamma; d_{\Gamma,a+k}) = (d_a + k)p - (d_a - 2)p - 2(p - 1) - 2 = kp.
\]

If \( a \not\in V(C) \), then by induction on \(|V(\Gamma)|\) and (12), we have
\[
R(\Gamma; d_{\Gamma,a+k}) = (d_a + k)p - d_ap = kp,
\]
as claimed. \(\square\)

Proposition A.5. Let \( \Gamma \) be a connected unicycle graph as above. If \( a \in V(T_i) \), \( b \in V(T_j) \), then we have
\[
R(\Gamma; d_{\Gamma,a+k,b+\ell}) = -k\ell \cdot d(i, j)^2 + k\ell p \cdot d(a, b) + (k + \ell)p.
\]

Proof. We use induction on \( d(a, b) \). When \( d(a, b) = 0 \), it has been proved in Proposition A.4. Let \( d(a, C) \geq 2 \). Then
\[
R(\Gamma; d_{\Gamma,a+k,b+\ell}) = (d_a + k)(-\ell d(i, j)^2 + \ell p(d(a, b) - 1) + (\ell + 1)p)
- (d_a - 1)(-\ell d(i, j)^2 + \ell p(d(a, b) - 1) + (\ell + 1)p)
- (-\ell d(i, j)^2 + \ell p(d(a, b) - 2) + (\ell + 1)p)
= -k\ell \cdot d(i, j)^2 + k\ell p \cdot d(a, b) + (k + \ell)p.
\]
The remaining cases can be verified similarly. We omit the details. \(\square\)

References