GEOMETRIC ANALYSIS

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1. The Sard theorem for mappings into ℓ^∞

In the theory of countably rectifiable metric spaces it is important to be able to verify whether the image of a Lipschitz mapping $f : \mathbb{R}^n \supset E \to X$ into a metric space satisfies $\mathcal{H}^n(f(E)) = 0$. The next result shows how to reduce this problem to the case of mappings into $\mathbb{R}^n$.

Definition 1.1. Let $f : Z \to X$ be a mapping between metric spaces and let $y_1, \ldots, y_n \in X$. The mapping $g : Z \to \mathbb{R}^n$ defined by

$$g(x) = (d(f(x), y_1), \ldots, d(f(x), y_n)),$$

where d denotes the metric in X, is called the *projection of f associated with points $y_1, \ldots, y_n$*.

Note that $\pi : X \to \mathbb{R}^n$, $\pi(y) = (d(y, y_1), \ldots, d(y, y_n))$ is Lipschitz. Hence if f is Lipschitz, then $g = \pi \circ f$ is Lipschitz too.

Theorem 1.2. *Let X be a metric space, let $E \subset \mathbb{R}^n$ be measurable, and let $f : E \to X$ be a Lipschitz mapping. Then the following statements are equivalent:*

1. $\mathcal{H}^n(f(E)) = 0$;
2. *For any Lipschitz mapping $\varphi : X \to \mathbb{R}^n$, we have $\mathcal{H}^n(\varphi(f(E))) = 0$;*
3. *For any collection of distinct points $\{y_1, y_2, \ldots, y_n\} \subset X$, the associated projection $g : E \to \mathbb{R}^n$ of f satisfies $\mathcal{H}^n(g(E)) = 0$;*
4. *For any collection of distinct points $\{y_1, y_2, \ldots, y_n\} \subset X$, the associated projection $g : E \to \mathbb{R}^n$ of f satisfies $\operatorname{rank}(\operatorname{ap} Dg(x)) < n$ for $\mathcal{H}^n$-a.e. $x \in E$.*

Remark 1.3. It follows from the proof that in conditions (3) and (4) we do not have to consider all families $\{y_1, y_2, \ldots, y_n\} \subset X$ of distinct points, but it suffices to consider such families with points y_i taken from a given countable and dense subset of $f(E)$.

The implications from (1) to (2) and from (2) to (3) are obvious. The equivalence between (3) and (4) easily follows from the change of variables formula (Theorem **??**): if $g : \mathbb{R}^n \supset E \to \mathbb{R}^n$ is Lipschitz, then

$$\int_E |J_g(x)|\, d\mathcal{H}^n(x) = \int_{g(E)} N_g(y,E)\, d\mathcal{H}^n(y). \tag{1.1}$$

Therefore, it remains to prove the implication (4) to (1) which is the most difficult part of the theorem. We will deduce it from another result which deals with Lipschitz mappings into ℓ^∞, see Theorem 1.5.

Remark 1.4. [1] In general it may happen for a subset $A \subset X$ that $\mathcal{H}^n(A) > 0$, but for all Lipschitz mappings $\varphi : X \to \mathbb{R}^n$, $\mathcal{H}^n(\varphi(A)) = 0$. For example the Heisenberg group[2] $\mathbb{H}^k$ satisfies $\mathcal{H}^{2k+2}(\mathbb{H}^k) = \infty$, but $\mathcal{H}^{2k+2}(\varphi(\mathbb{H}^k)) = 0$ for all Lipschitz mappings $\varphi : \mathbb{H}^k \to \mathbb{R}^{2k+2}$. Hence the implication from (2) to (1) has to use in an essential way that the assumption that $A = f(E)$ is a Lipschitz image of a Euclidean set. Since the condition (2) is satisfied for $X = \mathbb{H}^k$ with $n = 2k+2$, we conclude that $\mathbb{H}^k$ is purely $(2k+2)$-unrectifiable.

Let $f = (f_1, f_2, \ldots) : \mathbb{R}^n \supset E \to \ell^\infty$ be an L-Lipschitz mappings. Then the components $f_i : E \to \mathbb{R}$ are also L-Lipschitz. Hence for $\mathcal{H}^n$-almost all points $x \in E$, all functions f_i, $i \in \mathbb{N}$ are approximately differentiable at $x \in E$. We define the approximate derivative of f componentwise

$$\operatorname{ap} Df(x) = (\operatorname{ap} Df_1(x), \operatorname{ap} Df_2(x), \ldots).$$

For each $i \in \mathbb{N}$, $\operatorname{ap} Df_i(x)$ is a vector in $\mathbb{R}^n$ with component bounded by L. Hence $\operatorname{ap} Df(x)$ can be regarded as an $n \times \infty$ matrix of real numbers bounded by L, i.e.

$$\operatorname{ap} Df(x) \in (\ell^\infty)^n, \qquad \|\operatorname{ap} Df\|_\infty \leq L,$$

where the norm in $(\ell^\infty)^n$ is defined as the supremum over all entries in the $n \times \infty$ matrix. The meaning of the rank of the $n \times \infty$ matrix $\operatorname{ap} Df(x)$ is clear. It is the dimension of the linear subspace of $\mathbb{R}^n$ spanned by the vectors $\operatorname{ap} Df_i(x)$, $i \in \mathbb{N}$. Hence $\operatorname{rank}(\operatorname{ap} Df(x)) \leq n$ a.e.

[1] See Section 11.5 in DAVID, G., SEMMES, S.: *Fractured fractals and broken dreams. Self-similar geometry through metric and measure.* Oxford Lecture Series in Mathematics and its Applications, 7. The Clarendon Press, Oxford University Press, New York, 1997.

[2] Do not worry if you do not know what the Heisenberg groups are. This is just an example that will not be used in what is to follow.

The next theorem is the main step in the proof of the remaining implication (4) to (1) of Theorem 1.2.

Theorem 1.5. *Let $E \subset \mathbb{R}^n$ be measurable and let $f : E \to \ell^\infty$ be a Lipschitz mapping. Then $\mathcal{H}^n(f(E)) = 0$ if and only if $\operatorname{rank}(\operatorname{ap} Df(x)) < n$, $\mathcal{H}^n$-a.e. in E.*

Before we prove this result we will show how to use it to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. As we already pointed out, it remains to prove the implication from (4) to (1). Although we do not assume that X is separable, the image $f(E) \subset X$ is separable and hence it can be isometrically embedded into ℓ^∞ via the Kuratowski embedding (Theorem **??**). More precisely let $\{y_i\}_{i=1}^\infty \subset f(E)$ be a dense subset and let $y_0 \in f(E)$. Then

$$f(E) \ni y \mapsto \kappa(y) = \{d(y, y_i) - d(y_i, y_0)\}_{i=1}^\infty \in \ell^\infty$$

is an isometric embedding of $f(E)$ into ℓ^∞. Clearly

$$\mathcal{H}^n_d(f(E)) = \mathcal{H}^n_{\ell^\infty}((\kappa \circ f)(E)),$$

where subscripts indicate metrics with respect to which we define the Hausdorff measures. It remains to prove that $\mathcal{H}^n_{\ell^\infty}((\kappa \circ f)(E)) = 0$. Since

$$(\kappa \circ f)(x) = \{d(f(x), y_i) - d(y_i, y_0)\}_{i=1}^\infty$$

it easily follows from the assumptions that

$$\operatorname{rank}(\operatorname{ap} D(\kappa \circ f)) < n \quad \mathcal{H}^n\text{-a.e. in } E.$$

Hence (1) follows from Theorem 1.5. $\square$

Thus it remains to prove Theorem 1.5. Before doing this let us make some comments explaining why it is not easy. Theorem 1.5 is related to the Sard theorem for Lipschitz mappings (Theorem **??**) which states that if $f : \mathbb{R}^n \to \mathbb{R}^m$, $m \geq n$ is Lipschitz, then

$$\mathcal{H}^n(f(\{x \in \mathbb{R}^n : \operatorname{rank} Df(x) < n\})) = 0.$$

The standard proof presented earlier is based on the observation that if $\operatorname{rank} Df(x) < n$, then for any $\varepsilon > 0$ there is $r > 0$ such that

$$|f(z) - f(x) - Df(x)(z - x)| < \varepsilon r \quad \text{for } z \in B(x, r)$$

and hence

$$\operatorname{dist}(f(z), W_x) \leq \varepsilon r \quad \text{for } z \in B(x,r),$$

where $W_x = f(x) + Df(x)(\mathbb{R}^n)$ is an affine subspace of $\mathbb{R}^m$ of dimension less than or equal to $n-1$. That means $f(B(x,r))$ is contained in a thin neighborhood of an ellipsoid of dimension no greater than $n-1$ and hence we can cover it by $C(L/\varepsilon)^{n-1}$ balls of radius $C\varepsilon r$, where L is the Lipschitz constant of f. Now we use the $5r$-covering lemma to estimate the Hausdorff content of the image of the critical set.

The proof described here employs the fact that f is Frechet differentiable and hence this argument *cannot* be applied to the case of mappings into ℓ^∞, because in general Lipschitz mappings into ℓ^∞ are not Frechet differentiable, i.e. in general the image of $f(B(x,r)\cap E)$ is not well approximated by the tangent mapping $\operatorname{ap} Df(x)$. To overcome this difficulty we need to investigate the structure of the set $\{\operatorname{ap} Df(x) < n\}$ using arguments employed in the proof of the general case of the Sard theorem for C^k mappings that will be presented in Section **??**. In particular we will need to use a version of the implicit function theorem.

In the proof of Theorem 1.5 we will also need the following result which is of independent interest.

Proposition 1.6. *Let $D \subset \mathbb{R}^n$ be a bounded and convex set with non-empty interior and let $f : D \to \ell^\infty$ be an L-Lipschitz mapping. Then*

$$\operatorname{diam}(f(D)) \leq C(n)L\frac{(\operatorname{diam} D)^n}{\mathcal{H}^n(D)}\,\mathcal{H}^n(D\setminus A)^{1/n}$$

where

$$A = \{x \in D : \ Df(x) = 0\}.$$

In particular if D is a cube or a ball, then

$$\operatorname{diam}(f(D)) \leq C(n)L\mathcal{H}^n(D\setminus A)^{1/n} \tag{1.2}$$

Proof. We will need two well known facts.

Lemma 1.7. *If $E \subset \mathbb{R}^n$ is measurable, then*

$$\int_E \frac{dy}{|x-y|^{n-1}} \leq C(n)\mathcal{H}^n(E)^{1/n}.$$

Proof. Let $B = B(x,r) \subset \mathbb{R}^n$ be a ball such that $\mathcal{H}^n(B) = \mathcal{H}^n(E)$. Then

$$\int_E \frac{dy}{|x-y|^{n-1}} \leq \int_B \frac{dy}{|x-y|^{n-1}} = C(n)r = C'(n)\mathcal{H}^n(E)^{1/n}.$$

□

The next lemma will be proved in Section **??**.

Lemma 1.8. *If $D \subset \mathbb{R}^n$ is a bounded and convex set with non-empty interior and if $u : D \to \mathbb{R}$ is Lipschitz continuous, then*

$$|u(x) - u_D| \leq \frac{(\operatorname{diam} D)^n}{n\mathcal{H}^n(D)} \int_D \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy \quad \textit{for all } x \in D,$$

where

$$u_D = \frac{1}{\mathcal{H}^n(D)} \int_D u(x)\, dx.$$

Now we can complete the proof of Proposition 1.6. If $Df(x) = 0$, then $\nabla f_i(x) = 0$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ we have

$$\begin{aligned} |f_i(x) - f_{iD}| &\leq \frac{(\operatorname{diam} D)^n}{n\mathcal{H}^n(D)} \int_D \frac{|\nabla f_i(y)|}{|x-y|^{n-1}}\, dy \leq \frac{L(\operatorname{diam} D)^n}{n\mathcal{H}^n(D)} \int_{D\setminus A} \frac{dy}{|x-y|^{n-1}} \\ &\leq C(n)L\frac{(\operatorname{diam} D)^n}{\mathcal{H}^n(D)}\, \mathcal{H}^n(D\setminus A)^{1/n}. \end{aligned}$$

Hence for all $x, y \in D$

$$|f_i(x) - f_i(y)| \leq |f_i(x) - f_{iD}| + |f_i(y) - f_{iD}| \leq 2C(n)L\frac{(\operatorname{diam} D)^n}{\mathcal{H}^n(D)}\, \mathcal{H}^n(D\setminus A)^{1/n}.$$

Taking supremum over $i \in \mathbb{N}$ yields

$$\|f(x) - f(y)\|_\infty \leq 2C(n)L\frac{(\operatorname{diam} D)^n}{\mathcal{H}^n(D)}\, \mathcal{H}^n(D\setminus A)^{1/n}$$

and the result follows upon taking supremum over all $x, y \in D$. □

Proof of Theorem 1.5. The implication from left to right is easy. Suppose that $\mathcal{H}^n(f(E)) = 0$. For any positive integers $i_1 < i_2 < \ldots < i_n$ the projection

$$\ell^\infty \ni (y_1, y_2, \ldots) \to (y_{i_1}, y_{i_2}, \ldots, y_{i_n}) \in \mathbb{R}^n$$

is Lipschitz continuous and hence the set

$$(f_{i_1}, \ldots, f_{i_n})(E) \subset \mathbb{R}^n$$

has $\mathcal{H}^n$-measure zero. It follows from the change of variables formula (1.1) that the matrix $[\partial f_{i_j}/\partial x_\ell]_{j,\ell=1}^n$ of approximate partial derivatives has rank less than n almost everywhere in E. Since this is true for any choice of $i_1 < i_2 < \ldots < i_n$ we conclude that $\operatorname{rank}(\operatorname{ap} Df(x)) < n$ a.e. in E.

Suppose now that $\operatorname{rank}(\operatorname{ap} Df(x)) < n$ a.e. in E. We need to prove that $\mathcal{H}^n(f(E)) = 0$. This implication is more difficult. Since $f_i : E \to \mathbb{R}$ is Lipschitz continuous, for any $\varepsilon > 0$ there is $g_i \in C^1(\mathbb{R}^n)$ such that

$$\mathcal{H}^n(\{x \in E : \ f_i(x) \neq g_i(x)\}) < \varepsilon/2^i.$$

Moreover $\operatorname{ap} Df_i(x) = Dg_i(x)$ for almost all points of the set where $f_i = g_i$ (Theorem **??**(d)). Hence there is a measurable set $F \subset E$ such that $\mathcal{H}^n(E \setminus F) < \varepsilon$ and

$$f = g \quad \text{and} \quad \operatorname{ap} Df(x) = Dg(x) \quad \text{in } F$$

where

$$g = (g_1, g_2, \ldots), \quad Dg = (Dg_1, Dg_2, \ldots).$$

It suffices to prove that $\mathcal{H}^n(f(F)) = 0$, because we can exhaust E with sets F up to a subset of measure zero and f maps sets of measure zero to sets of measure zero. Let

$$\tilde{F} = \{x \in F : \ \operatorname{rank}(\operatorname{ap} Df(x)) = \operatorname{rank} Dg(x) < n\}.$$

Since $\mathcal{H}^n(F \setminus \tilde{F}) = 0$, it suffices to prove that $\mathcal{H}^n(f(\tilde{F})) = 0$. For $0 \leq j \leq n-1$ let

$$K_j = \{x \in \tilde{F} : \operatorname{rank} Dg(x) = j\}.$$

Since $\tilde{F} = \bigcup_{j=0}^{n-1} K_j$, it suffices to prove that $\mathcal{H}^n(f(K_j)) = 0$ for any $0 \leq j \leq n-1$. Again, by removing a subset of measure zero we can assume that all points of K_j are density points of K_j. To prove that $\mathcal{H}^n(f(K_j)) = 0$ we need to make a change of variables in $\mathbb{R}^n$, but only when $j \geq 1$.

If $x \in \mathbb{R}^n \setminus F$, the sequence $(g_1(x), g_2(x), \ldots)$ is not necessarily bounded. Let V be the linear space of all real sequences $(y_1, y_2, \ldots)$. Clearly $g : \mathbb{R}^n \to V$. We do not equip V with any metric structure. Note that $g|_F : F \to \ell^\infty \subset V$, because g coincides with f on F.

Lemma 1.9. *Let $1 \leq j \leq n-1$ and $x_0 \in K_j$. Then there exists a neighborhood $x_0 \in U \subset \mathbb{R}^n$, a diffeomorphism $\Phi : U \subset \mathbb{R}^n \to \Phi(U) \subset \mathbb{R}^n$, and a composition of a translation (by a vector from ℓ^∞) with a permutation of variables $\Psi \colon V \to V$ such that*

- $\Phi^{-1}(0) = x_0$ *and* $\Psi(g(x_0)) = 0$;
- *There is $\varepsilon > 0$ such that for $x = (x_1, x_2, \ldots, x_n) \in B(0, \varepsilon) \subset \mathbb{R}^n$ and $i = 1, 2, \ldots, j$,*

  $$\left(\Psi \circ g \circ \Phi^{-1}\right)_i (x) = x_i,$$

  i.e., $\Psi \circ g \circ \Phi^{-1}$ fixes the first j variables in a neighborhood of 0.

Proof. By precomposing g with a translation of $\mathbb{R}^n$ by the vector x_0 and postcomposing it with a translation of V by the vector $-g(x_0) = -f(x_0) \in \ell^\infty$ we may assume that $x_0 = 0$ and $g(x_0) = 0$. A certain $j \times j$ minor of $Dg(x_0)$ has rank j. By precomposing g with a permutation of j variables in $\mathbb{R}^n$ and postcomposing it with a permutation of j variables in V we may assume that

$$\operatorname{rank} \left[\frac{\partial g_m}{\partial x_\ell}(x_0) \right]_{1 \leq m, \ell \leq j} = j. \tag{1.3}$$

Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$H(x) = (g_1(x), \ldots, g_j(x), x_{j+1}, \ldots, x_n).$$

It follows from (1.3) that $J_H(x_0) \neq 0$ and hence H is a diffeomorphism in a neighborhood of $x_0 = 0 \in \mathbb{R}^n$. It suffices to observe that for all $i = 1, 2, \ldots, j$,

$$\left(g \circ H^{-1}\right)_i (x) = x_i.$$

$\square$

In what follows, by cubes we will mean cubes with edges parallel to the coordinate axes in $\mathbb{R}^n$. It suffices to prove that any point $x_0 \in K_j$ has a cubic neighborhood whose intersection with K_j is mapped onto a set of $\mathcal{H}^n$-measure zero. Since we can take cubic neighborhoods to be arbitrarily small, the change of variables from Lemma 1.9 allows us to assume that

$$K_j \subset (0,1)^n, \quad g_i(x) = x_i \quad \text{for } i = 1, 2, \ldots, j \text{ and } x \in [0,1]^n. \tag{1.4}$$

Indeed, according to Lemma 1.9 we can assume that $x_0 = 0$ and that g fixes the first j variables in a neighborhood of 0. The neighborhood can be very small, but a rescaling argument allows us to assume that it contains a unit cube Q around 0. Translating the cube we can assume that $Q = [0,1]^n$. If $x \in K_j$, since $\operatorname{rank} Dg(x) = j$ and g fixes the first j coordinates, the derivative of g in directions orthogonal to the first j coordinates equals zero at x, $\partial g_k(x)/\partial x_i = 0$ for $i = j+1, \ldots, n$ and any k.

Lemma 1.10. *Under the assumptions* (1.4) *there exists a constant $C = C(n) > 0$ such that for any integer $m \geq 1$, and every $x \in K_j$, there is a closed cube $Q_x \subset [0,1]^n$ with edge length d_x centered at x with the property that $f(K_j \cap Q_x) = g(K_j \cap Q_x)$ can be covered by m^j balls in ℓ^∞ each of radius $CLd_x m^{-1}$, where L is the Lipschitz constant of f.*

The theorem is an easy consequence of this lemma through a standard application of the $5r$-covering lemma (Theorem **??**); we used a similar argument in the proof of the Sard theorem. First of all observe that cubes with sides parallel to coordinate axes in $\mathbb{R}^n$ are balls with respect to the ℓ^∞_n metric

$$\|x-y\|_\infty = \max_{1\leq i\leq n} |x_i - y_i|.$$

Hence the $5r$-covering lemma applies to families of cubes in $\mathbb{R}^n$. By $5^{-1}Q$ we will denote a cube concentric with Q and with 5^{-1} times the diameter. The cubes $\{5^{-1}Q_x\}_{x\in K_j}$ form a covering of K_j. Hence we can select disjoint cubes $\{5^{-1}Q_{x_i}\}_{i=1}^\infty$ such that

$$K_j \subset \bigcup_{i=1}^\infty Q_{x_i}.$$

If d_i is the edge length of Q_{x_i}, then $\sum_{i=1}^\infty (5^{-1}d_i)^n \leq 1$, because the cubes $5^{-1}Q_{x_i}$ are disjoint and contained in $[0,1]^n$. Hence

$$\mathcal{H}^n_\infty(f(K_j)) \leq \sum_{i=1}^\infty \mathcal{H}^n_\infty(f(K_j\cap Q_{x_i})) \leq \sum_{i=1}^\infty m^j(CLd_i m^{-1})^n \leq 5^n C^n L^n m^{j-n}.$$

Since the exponent $j-k$ is negative, and m can be arbitrarily large we conclude that $\mathcal{H}^n_\infty(f(K_j)) = 0$ and hence $\mathcal{H}^n(f(K_j)) = 0$.

Thus it remains to prove Lemma 1.10.

Proof of Lemma 1.10. Various constants C in the proof below will depend on n only. Fix an integer $m\geq 1$. Let $x\in K_j$. Since every point in K_j is a density point of K_j, there is a closed cube $Q\subset [0,1]^n$ centered at x of edge length d such that

$$\mathcal{H}^n(Q\setminus K_j) < m^{-n}\mathcal{H}^n(Q) = m^{-n}d^n. \tag{1.5}$$

By translating the coordinate system in $\mathbb{R}^n$ we may assume that

$$Q = [0,d]^j \times [0,d]^{n-j}.$$

Each component of $f : Q\cap K_j \to \ell^\infty$ is an L-Lipschitz function. Extending each component to an L-Lipschitz function on Q results in an L-Lipschitz extension $\tilde{f} : Q\to \ell^\infty$. This is well known and easy to check.

Divide $[0,d]^j$ into m^j cubes with pairwise disjoint interiors, each of edge length $m^{-1}d$. Denote the resulting cubes by Q_ν, $\nu\in\{1,2,\ldots,m^j\}$. It remains to prove that

$$f((Q_\nu\times[0,d]^{n-j})\cap K_j) \subset \tilde{f}(Q_\nu\times[0,d]^{n-j})$$

is contained in a ball (in ℓ^∞) of radius $CLdm^{-1}$. It follows from (1.5) that

$$\mathcal{H}^n((Q_\nu \times [0,d]^{n-j}) \setminus K_j) \leq \mathcal{H}^n(Q \setminus K_j) < m^{-n}d^n.$$

Hence

$$\mathcal{H}^n((Q_\nu \times [0,d]^{n-j}) \cap K_j) > (m^{-j} - m^{-n})d^n.$$

This estimate and the Fubini theorem imply that there is $\rho \in Q_\nu$ such that

$$\mathcal{H}^{n-j}((\{\rho\} \times [0,d]^{n-j}) \cap K_j) > (1 - m^{j-n})d^{n-j}.$$

Hence

$$\mathcal{H}^{n-j}((\{\rho\} \times [0,d]^{n-j}) \setminus K_j) < m^{j-n}d^{n-j}.$$

It follows from (1.2) with n replaced by $n-j$ that

$$\text{diam}_{\ell^\infty}(\tilde{f}(\{\rho\} \times [0,d]^{n-j})) \leq CL\mathcal{H}^{n-j}(\{\rho\} \times [0,d]^{n-j}) \setminus K_j)^{1/(n-j)} \leq CLm^{-1}d. \tag{1.6}$$

Indeed, the rank of the derivative of g restricted to the slice $\{\rho\} \times [0,d]^{n-j}$ equals zero at the points of $(\{\rho\} \times [0,d]^{n-j}) \cap K_j$ and this derivative coincides a.e. with the approximate derivative of $\tilde{f}$ restricted to $(\{\rho\} \times [0,d]^{n-j}) \cap K_j$ which by the property of g must be zero as well.

Since the distance of any point in $Q_\nu \times [0,d]^{n-j}$ to $\{\rho\} \times [0,d]^{n-j}$ is bounded by $Cm^{-1}d$ and $\tilde{f}$ is L-Lipschitz, (1.6) implies that $\tilde{f}(Q_\nu \times [0,d]^{n-j})$ is contained in a ball of radius $CLdm^{-1}$, perhaps with a constant C bigger than that in (1.6). The proof is the lemma is complete. $\square$

This also completes the proof of Theorem 1.5. $\square$

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