Print your first and last name legibly above the line:

Calculus III

Professor Piotr Hajłasz

First Exam

February 18, 2015.

Problem	Possible points	Score
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
Total	100	100

Good job!
By the way, what is your name?
I asked to write the first and the last name <u>legibly</u>!!!

Problem 1. (20p=4×5p)

(a) Find the equation of the plane that passes through the points $P(1,1,1)$, $Q(1,2,3)$, $R(3,2,1)$.

$\overrightarrow{PQ} = \langle 1-1, 2-1, 3-1 \rangle = \langle 0,1,2 \rangle$, $\overrightarrow{PR} = \langle 3-1, 2-1, 1-1 \rangle = \langle 2,1,0 \rangle$

$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = \langle -2, 4, -2 \rangle$ is normal to the plane. Given normal $\langle -2, 4, -2 \rangle$ and the point $(1,1,1)$ on the plane, the equation is

$$-2(x-1) + 4(y-1) - 2(z-1) = 0$$

$$-2x + 2 + 4y - 4 - 2z + 2 = 0$$

$$\boxed{-2x + 4y - 2z = 0}$$

(b) Find the area of the triangle with the vertices $P(1,1,1)$, $Q(1,2,3)$, $R(3,2,1)$.

$$\text{Area} = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{(-2)^2 + 4^2 + (-2)^2}$$

$$= \frac{1}{2}\sqrt{24} = \boxed{\sqrt{6}}$$

(c) For what values of the parameter a are the planes $x+ay+2z=2015$ and $3x+(a+2)y+6z=1410$ parallel?

The normal vectors $\langle 1, a, 2\rangle$, $\langle 3, a+2, 6\rangle$ to the planes must be parallel. By comparing the x components of the vectors we see that the proportionality factor must be equal 3

$$3\langle 1, a, 2\rangle = \langle 3, a+2, 6\rangle$$

$$\langle 3, 3a, 6\rangle = \langle 3, a+2, 6\rangle$$

$$3a = a+2$$

$$2a = 2$$

$$\boxed{a=1}$$

(d) Find the equations of the line of intersection of the planes $x+y+z=1$ and $x-2y+3z=1$.

The vectors $\langle 1,1,1\rangle$, $\langle 1,-2,3\rangle$ are orthogonal to the planes so they are orthogonal to the line of intersection. Hence the cross product of the two vectors is parallel to the line

$$\langle 1,1,1\rangle \times \langle 1,-2,3\rangle = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5,-2,-3\rangle.$$

We need to find a point on the line. Such a point must satisfy both equations

$$\begin{cases} x+y+z=1 \\ x-2y+3z=1 \end{cases}$$

We guess and we see that $(x,y,z) = (1,0,0)$ satisfies both equations. The parametric equations of the line are

$$\vec{r}(t) = \langle 1,0,0\rangle + t\langle 5,-2,-3\rangle$$

$$\boxed{x = 1+5t,\ y=-2t,\ z=-3t}$$

Problem 2. (20p=2×10p)

(a) Show that the limit $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)+y^2}{x^2+y^2}$ does not exist.

$$f(t,0) = \frac{\sin(t\cdot 0)+0^2}{t^2+0^2} = 0 \xrightarrow{t\to 0} 0$$

$$f(0,t) = \frac{\sin(0\cdot t)+t^2}{0^2+t^2} = \frac{t^2}{t^2} = 1 \xrightarrow{t\to 0} 1$$

Hence the limit does not exist.

(b) Show that the limit exists and find it. $\lim_{(x,y)\to(0,0)} \frac{e^{xy}(x^2+y^2)+y^2\sin(xy^2)}{2(x^2+y^2)}$.

$$\frac{e^{xy}(x^2+y^2)+y^2\sin(xy^2)}{2(x^2+y^2)} =$$

$$\frac{1}{2}e^{xy \to 0} + \underbrace{\frac{y^2}{2(x^2+y^2)}}_{\leq 1/2} \sin(xy^2 \to 0) \longrightarrow$$

$$\xrightarrow{(x,y)\to(0,0)} \frac{1}{2}e^0 + 0 = \frac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}(x^2+y^2)+y^2\sin(xy^2)}{2(x^2+y^2)} = \boxed{\frac{1}{2}}$$

Problem 3. (20p=2×10p)
(a) Find the curvature of the helix $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$.

$$\vec{r}\,'(t) = \langle -3\sin t,\ 3\cos t,\ 4 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} =$$

$$= \sqrt{3^2(\sin^2 t + \cos^2 t) + 4^2} = \sqrt{3^2 + 4^2} = 5$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{5}\langle -3\sin t,\ 3\cos t,\ 4\rangle$$

$$\vec{T}\,'(t) = \frac{1}{5}\langle -3\cos t,\ -3\sin t,\ 0 \rangle$$

$$|\vec{T}\,'(t)| = \frac{1}{5}\sqrt{(-3\cos t)^2 + (-3\sin t)^2 + 0^2}$$

$$= \frac{1}{5}\sqrt{3^2(\cos^2 t + \sin^2 t)} = \frac{3}{5}$$

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{3/5}{5} = \boxed{\frac{3}{25}}$$

(b) Find the normal and the binormal vectors to the helix $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$.

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}, \quad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

From the part (a) we have

$$\vec{T}(t) = \frac{1}{5} \langle -3\sin t, 3\cos t, 4 \rangle$$

$$\vec{T}'(t) = \frac{1}{5} \langle -3\cos t, -3\sin t, 0 \rangle$$

$$|\vec{T}'(t)| = \frac{3}{5}$$

Hence

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{1/5}{3/5} \langle -3\cos t, -3\sin t, 0 \rangle$$

$$= \frac{1}{3} \langle -3\cos t, -3\sin t, 0 \rangle = \langle -\cos t, -\sin t, 0 \rangle$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} i & j & k \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \left\langle \frac{4}{5}\sin t, -\frac{4}{5}\cos t, \frac{3}{5}\sin^2 t + \frac{3}{5}\cos^2 t \right\rangle =$$

$$= \frac{1}{5} \langle 4\sin t, -4\cos t, 3 \rangle$$

$$\boxed{\begin{aligned} \vec{N}(t) &= \langle -\cos t, -\sin t, 0 \rangle \\ \vec{B}(t) &= \frac{1}{5} \langle 4\sin t, -4\cos t, 3 \rangle \end{aligned}}$$

Problem 4. (20p=2×10p)
(a) Find ∇z where z is a function of variables x and y implicitly defined by the equation

$$xyz = 1 + \pi^3 + \cos(x + y + z).$$

$$\nabla z = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle$$

$$\frac{\partial}{\partial x}(xyz) = \frac{\partial}{\partial x}\left(1+\pi^3+\cos(x+y+z)\right)$$

$$yz + xy\frac{\partial z}{\partial x} = -\sin(x+y+z)\left(1+\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial z}{\partial x}\left(xy+\sin(x+y+z)\right) = -yz - \sin(x+y+z)$$

$$\frac{\partial z}{\partial x} = -\frac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}.$$

By th symmetry between variables x and y

$$\frac{\partial z}{\partial y} = -\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}$$

$$\nabla z = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle =$$

$$= \boxed{-\frac{\langle yz+\sin(x+y+z), xz+\sin(x+y+z)\rangle}{xy+\sin(x+y+z)}}$$

(b) Find the equation of the tangent plane to the surface $xyz = 1 + \pi^3 + \cos(x + y + z)$ at the point (π, π, π). **Hint:** *Use the results of the part (a).*

$$\pi \cdot \pi \cdot \pi = 1 + \pi^3 + \cos(\pi + \pi + \pi)$$
$$\pi^3 = 1 + \pi^3 + \cos(3\pi)$$
$$\pi^3 = 1 + \pi^3 - 1$$
$$\pi^3 = \pi^3 \quad \checkmark$$

so the point (π, π, π) is on the surface

1st method

The tangent plane equation is

$$z - z_0 = \frac{\partial z}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial z}{\partial y}(x_0, y_0)(y - y_0).$$

From the part (a) we have

$$\left\langle \frac{\partial z}{\partial x}(\pi, \pi), \frac{\partial z}{\partial y}(\pi, \pi) \right\rangle = -\frac{\langle \pi^2 + \sin(3\pi), \pi^2 + \sin(3\pi) \rangle}{\pi^2 + \sin(3\pi)}$$

$$= -\frac{\langle \pi^2, \pi^2 \rangle}{\pi^2} = \langle -1, -1 \rangle$$

$$z - \pi = -1 \cdot (x - \pi) + (-1)(y - \pi)$$

$$z - \pi = -x + \pi - y + \pi$$

$$\boxed{x + y + z = 3\pi}$$

2nd method

$$\underbrace{xyz - \cos(x+y+z)}_{F(x,y,z)} = 1+\pi^2$$

$\nabla F(\pi,\pi,\pi)$ is orthogonal to the surface at (π,π,π).

$$\nabla F = \langle yz + \sin(x+y+z),\ xz + \sin(x+y+z),\ xy + \sin(x+y+z) \rangle$$

$$\nabla F(\pi,\pi,\pi) = \langle \pi^2 + \sin(3\pi),\ \pi^2 + \sin(3\pi),\ \pi^2 + \sin(3\pi) \rangle$$

$$= \langle \pi^2, \pi^2, \pi^2 \rangle.$$

The equation of the tangent plane is

$$\pi^2(x-\pi) + \pi^2(y-\pi) + \pi^2(z-\pi) = 0$$

Divide by π^2:

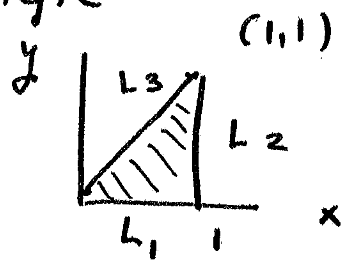
$$x - \pi + y - \pi + z - \pi = 0$$

$$\boxed{x+y+z = 3\pi}$$

Problem 5. (20p) Find the absolute maximum and minimum values of

$$x^3 - y^3 + 6xy, \quad 0 \le x \le 1, \quad 0 \le y \le x.$$

The function $f(x,y) = x^3 - y^3 + 6xy$ is defined on the triangle



Critical points in the interior

$$\begin{cases} f_x = 3x^2 + 6y = 0 \\ f_y = -3y^2 + 6x = 0 \end{cases}$$

In the interior of the triangle $y > 0$
so $f_x = 3x^2 + 6y > 0$ and hence there
are no critical points in the interior.

On L_1 $\quad 0 \le x \le 1, \; y = 0$

$$f(x,y) = f(x,0) = x^3, \quad 0 \le x \le 1$$

minimum at $x=0$, maximum at $x=1$

$$\boxed{f(0,0) = 0}, \quad \boxed{f(1,0) = 1}$$

On L_2 $\quad x = 1, \; 0 \le y \le 1$

$$f(x,y) = f(1,y) = 1 - y^3 + 6y, \quad 0 \le y \le 1$$

We have to investigate the function

$$g(y) = 1 - y^3 + 6y, \quad 0 \le y \le 1$$

Critical points

$$g'(y) = -3y^2 + 6 = 3(2 - y^2)$$

$$= 3(\sqrt{2} - y)(\sqrt{2} + y) = 0$$

The roots are $\pm\sqrt{2}$ and they are <u>not</u> in the interval $0 \le y \le 1$.

The function g has no critical points in the interval $[0,1]$ and we are left with the endpoints

$\boxed{f(1,0) = g(0) = 1}$, $\boxed{f(1,1) = g(1) = 6}$

<u>On L_3</u> $0 \le x \le 1$, $y = x$

$f(x,y) = f(x,x) = x^3 - x^3 + 6 \cdot x \cdot x = 6x^2, \, 0 \le x \le 1$

The function is increasing so it has extreme values at the end points

$\boxed{f(0,0) = 0}$, $\boxed{f(1,1) = 6}$

The absolute minimum and maximum are among the values

$f(0,0) = 0$, $f(1,0) = 1$, $f(1,1) = 6$

Hence

$$\boxed{\begin{array}{ll} f(0,0) = 0 & \text{— absolute minimum} \\ f(1,1) = 6 & \text{— absolute maximum} \end{array}}$$