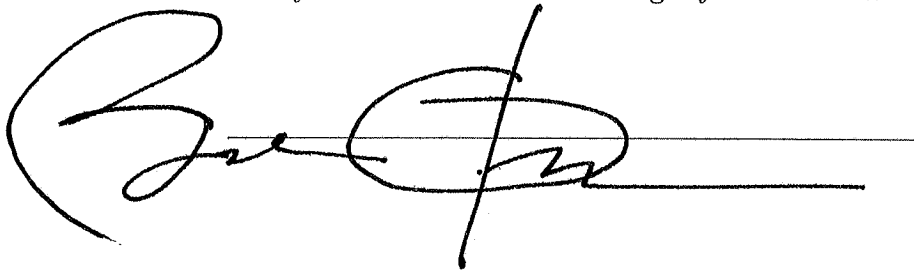


Print your first and last name legibly above the line:



Calculus III

Professor Piotr Hajlasz

First Exam

February 18, 2015.

Problem	Possible points	Score
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
Total	100	100

Good job!
By the way, what is your name?
I asked to write the first and
the last name legibly !!!

Problem 1. (20p=4×5p)

(a) Find the equation of the plane that passes through the points $P(1, 1, 1)$, $Q(1, 2, 3)$, $R(3, 2, 1)$.

$$\vec{PQ} = \langle 1-1, 2-1, 3-1 \rangle = \langle 0, 1, 2 \rangle, \quad \vec{PR} = \langle 3-1, 2-1, 1-1 \rangle = \langle 2, 1, 0 \rangle$$

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = \langle -2, 4, -2 \rangle$ is normal to the plane. Given normal $\langle -2, 4, -2 \rangle$ and the point $(1, 1, 1)$ on the plane, the equation is

$$-2(x-1) + 4(y-1) - 2(z-1) = 0$$

$$-2x + 2 + 4y - 4 - 2z + 2 = 0$$

$$\boxed{-2x + 4y - 2z = 0}$$

(b) Find the area of the triangle with the vertices $P(1, 1, 1)$, $Q(1, 2, 3)$, $R(3, 2, 1)$.

$$\text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{(-2)^2 + 4^2 + (-2)^2}$$

$$= \frac{1}{2} \sqrt{24} = \boxed{\sqrt{6}}$$

(c) For what values of the parameter a are the planes $x + ay + 2z = 2015$ and $3x + (a+2)y + 6z = 1410$ parallel?

The normal vectors $\langle 1, a, 2 \rangle$, $\langle 3, a+2, 6 \rangle$ to the planes must be parallel. By comparing the x components of the vectors we see that the proportionality factor must be equal 3

$$3\langle 1, a, 2 \rangle = \langle 3, a+2, 6 \rangle$$

$$\langle 3, 3a, 6 \rangle = \langle 3, a+2, 6 \rangle$$

$$3a = a + 2$$

$$2a = 2$$

$$\boxed{a = 1}$$

(d) Find the equations of the line of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

The vectors $\langle 1, 1, 1 \rangle$, $\langle 1, -2, 3 \rangle$ are orthogonal to the planes so they are orthogonal to the line of intersection. Hence the cross product of the two vectors is parallel to the line

$$\langle 1, 1, 1 \rangle \times \langle 1, -2, 3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5, -2, -3 \rangle.$$

We need to find a point on the line. Such a point must satisfy both equations

$$\begin{cases} x + y + z = 1 \\ x - 2y + 3z = 1 \end{cases}$$

We guess and we see that $(x, y, z) = (1, 0, 0)$ satisfies both equations. The parametric equations of the line are

$$\vec{r}(t) = \langle 1, 0, 0 \rangle + t\langle 5, -2, -3 \rangle$$

$$\boxed{x = 1 + 5t, y = -2t, z = -3t}$$

Problem 2. (20p=2×10p)

(a) Show that the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy) + y^2}{x^2 + y^2}$ does not exist.

$$f(t, 0) = \frac{\sin(t \cdot 0) + 0^2}{t^2 + 0^2} = 0 \xrightarrow{t \rightarrow 0} 0$$

$$f(0, t) = \frac{\sin(0 \cdot t) + t^2}{0^2 + t^2} = \frac{t^2}{t^2} = 1 \xrightarrow{t \rightarrow 0} 1$$

Hence the limit does not exist.

(b) Show that the limit exists and find it. $\lim_{(x,y) \rightarrow (0,0)} \frac{e^{xy}(x^2 + y^2) + y^2 \sin(xy^2)}{2(x^2 + y^2)}$.

$$\frac{e^{xy}(x^2 + y^2) + y^2 \sin(xy^2)}{2(x^2 + y^2)} =$$

$$\frac{1}{2} e^{xy} + \frac{y^2}{2(x^2 + y^2)} \sin(xy^2) \rightarrow$$

$$\xrightarrow{(x,y) \rightarrow (0,0)} \frac{1}{2} e^0 + 0 = \frac{1}{2}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^{xy}(x^2 + y^2) + y^2 \sin(xy^2)}{2(x^2 + y^2)} = \boxed{\frac{1}{2}}$$

Problem 3. (20p=2×10p)

(a) Find the curvature of the helix $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$.

$$\vec{r}'(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle$$

$$\begin{aligned} |\vec{r}'(t)| &= \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + 4^2} = \\ &= \sqrt{3^2 (\sin^2 t + \cos^2 t) + 4^2} = \sqrt{3^2 + 4^2} = 5 \end{aligned}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{5} \langle -3 \sin t, 3 \cos t, 4 \rangle$$

$$\vec{T}'(t) = \frac{1}{5} \langle -3 \cos t, -3 \sin t, 0 \rangle$$

$$\begin{aligned} |\vec{T}'(t)| &= \frac{1}{5} \sqrt{(-3 \cos t)^2 + (-3 \sin t)^2 + 0^2} \\ &= \frac{1}{5} \sqrt{3^2 (\cos^2 t + \sin^2 t)} = \frac{3}{5} \end{aligned}$$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{3/5}{5} = \boxed{\frac{3}{25}}$$

(b) Find the normal and the binormal vectors to the helix $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$.

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}, \quad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

From the part (a) we have

$$\vec{T}(t) = \frac{1}{5} \langle -3 \sin t, 3 \cos t, 4 \rangle$$

$$\vec{T}'(t) = \frac{1}{5} \langle -3 \cos t, -3 \sin t, 0 \rangle$$

$$|\vec{T}'(t)| = \frac{3}{5}$$

Hence

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{1/5}{3/5} \langle -3 \cos t, -3 \sin t, 0 \rangle$$

$$= \frac{1}{3} \langle -3 \cos t, -3 \sin t, 0 \rangle = \langle -\cos t, -\sin t, 0 \rangle$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} i & j & k \\ -\frac{3}{5} \sin t & \frac{3}{5} \cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \left\langle \frac{4}{5} \sin t, -\frac{4}{5} \cos t, \frac{3}{5} \sin^2 t + \frac{3}{5} \cos^2 t \right\rangle \Rightarrow$$

$$= \frac{1}{5} \langle 4 \sin t, -4 \cos t, 3 \rangle$$

$$\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

$$\vec{B}(t) = \frac{1}{5} \langle 4 \sin t, -4 \cos t, 3 \rangle$$

Problem 4. (20p=2×10p)

(a) Find ∇z where z is a function of variables x and y implicitly defined by the equation

$$xyz = 1 + \pi^3 + \cos(x+y+z).$$

$$\nabla z = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle$$

$$\frac{\partial}{\partial x} (xyz) = \frac{\partial}{\partial x} (1 + \pi^3 + \cos(x+y+z))$$

$$yz + xy \frac{\partial z}{\partial x} = -\sin(x+y+z) \left(1 + \frac{\partial z}{\partial x}\right)$$

$$\frac{\partial z}{\partial x} (xy + \sin(x+y+z)) = -yz - \sin(x+y+z)$$

$$\frac{\partial z}{\partial x} = - \frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$

By the symmetry between variables x and y

$$\frac{\partial z}{\partial y} = - \frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$

$$\nabla z = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle =$$

$$= \left\langle - \frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}, - \frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)} \right\rangle$$

(b) Find the equation of the tangent plane to the surface $xyz = 1 + \pi^3 + \cos(x + y + z)$ at the point (π, π, π) . Hint: Use the results of the part (a).

$$\pi \cdot \pi \cdot \pi = 1 + \pi^3 + \cos(\pi + \pi + \pi)$$

$$\pi^3 = 1 + \pi^3 + \cos(3\pi)$$

$$\pi^3 = 1 + \pi^3 - 1$$

$$\pi^3 = \pi^3 \quad \checkmark$$

so the point (π, π, π) is on the surface

1st method

The tangent plane equation is

$$z - z_0 = \frac{\partial z}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial z}{\partial y}(x_0, y_0)(y - y_0)$$

From the part (a) we have

$$\left\langle \frac{\partial z}{\partial x}(\pi, \pi), \frac{\partial z}{\partial y}(\pi, \pi) \right\rangle = - \frac{\langle \pi^2 + \sin(3\pi), \pi^2 + \sin(3\pi) \rangle}{\pi^2 + \sin(3\pi)}$$

$$= - \frac{\langle \pi^2, \pi^2 \rangle}{\pi^2} = \langle -1, -1 \rangle$$

$$z - \pi = -1 \cdot (x - \pi) + (-1)(y - \pi)$$

$$z = \pi = -x + \pi - y + \pi$$

$$\boxed{x + y + z = 3\pi}$$

2nd method

$$\underbrace{xyz - \cos(x+y+z)}_{F(x,y,z)} = 1 + \pi^2$$

$\nabla F(\pi, \pi, \pi)$ is orthogonal to the surface at (π, π, π) .

$$\nabla F = \langle yz + \sin(x+y+z), xz + \sin(x+y+z), xy + \sin(x+y+z) \rangle$$

$$\begin{aligned}\nabla F(\pi, \pi, \pi) &= \langle \pi^2 + \sin(3\pi), \pi^2 + \sin(3\pi), \pi^2 + \sin(3\pi) \rangle \\ &= \langle \pi^2, \pi^2, \pi^2 \rangle.\end{aligned}$$

The equation of the tangent plane is

$$\pi^2(x-\pi) + \pi^2(y-\pi) + \pi^2(z-\pi) = 0$$

Divide by π^2 :

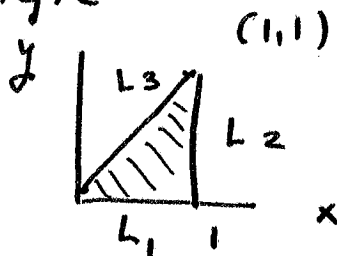
$$x - \pi + y - \pi + z - \pi = 0$$

$$\boxed{x + y + z = 3\pi}$$

Problem 5. (20p) Find the absolute maximum and minimum values of

$$x^3 - y^3 + 6xy, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq x.$$

The function $f(x,y) = x^3 - y^3 + 6xy$ is defined on the triangle



Critical points in the interior

$$\begin{cases} f_x = 3x^2 + 6y = 0 \\ f_y = -3y^2 + 6x = 0 \end{cases}$$

In the interior of the triangle $y > 0$
 so $f_x = 3x^2 + 6y > 0$ and hence there
 are no critical points in the interior.

On L_1 $0 \leq x \leq 1, y = 0$

$$f(x,y) = f(x,0) = x^3, \quad 0 \leq x \leq 1$$

minimum at $x=0$, maximum at $x=1$

$$\boxed{f(0,0) = 0}, \quad \boxed{f(1,0) = 1}$$

On L_2 $x = 1, 0 \leq y \leq 1$

$$f(x,y) = f(1,y) = 1 - y^3 + 6y, \quad 0 \leq y \leq 1$$

We have to investigate the function

$$g(y) = 1 - y^3 + 6y, \quad 0 \leq y \leq 1$$

Critical points

$$\begin{aligned} g'(y) &= -3y^2 + 6 = 3(2 - y^2) \\ &= 3(\sqrt{2} - y)(\sqrt{2} + y) = 0 \end{aligned}$$

The roots are $\pm \sqrt{2}$ and they are not in the interval $0 \leq y \leq 1$.

The function g has no critical points in the interval $[0, 1]$ and we are left with the endpoints

$$\boxed{f(1, 0) = g(0) = 1}, \quad \boxed{f(1, 1) = g(1) = 6}$$

On L_3 $0 \leq x \leq 1, y = x$

$$f(x, y) = f(x, x) = x^3 - x^3 + 6 \cdot x \cdot x = 6x^2, \quad 0 \leq x \leq 1$$

The function is increasing so it has extreme values at the endpoints

$$\boxed{f(0, 0) = 0}, \quad \boxed{f(1, 1) = 6}$$

The absolute minimum and maximum are among the values

$$f(0, 0) = 0, \quad f(1, 0) = 1, \quad f(1, 1) = 6$$

Hence

$$\boxed{\begin{array}{l} f(0, 0) = 0 \quad - \text{absolute minimum} \\ f(1, 1) = 6 \quad - \text{absolute maximum} \end{array}}$$