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Calculus III

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First Exam

October 10, 2016.

Problem	Possible points	Score
1	20	
2	10	
3	10	
4	20	
5	10	
6	10	
7	20	
Total	100	

Problem 1. (20p)

(a) For what values of a are the planes $x + 2y = 3 - z$ and $x + ay - 2z = 5$ orthogonal?

$x + 2y + z = 3, \quad \vec{n_1} = \langle 1, 2, 1 \rangle$

$x + ay - 2z = 5, \quad \vec{n_2} = \langle 1, a, -2 \rangle$

$\vec{n_1} \cdot \vec{n_2} = 0$

$\langle 1, 2, 1 \rangle \cdot \langle 1, a, -2 \rangle = 0$

$1 + 2a - 2 = 0$

$2a = 1$

$\boxed{a = \frac{1}{2}}$

(b) Find the equation of a plane passing through the point $(1, 2, 3)$ that is parallel to the plane $x = 2y + 3z$.

$x - 2y - 3z = 0$

point $(1, 2, 3)$

normal $\langle 1, -2, -3 \rangle$

equation

$\boxed{1 \cdot (x-1) - 2(y-2) - 3(z-3) = 0}$

(c) Find the equation of a plane passing through the points $A(1,1,1)$, $B(0,2,5)$, $C(-1,0,1)$

$\vec{AB} = \langle -1, 1, 4 \rangle$, $\quad \vec{AC} = \langle -2, -1, 0 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 1 & 4 \\ -2 & -1 & 0 \end{vmatrix} = \langle 4, -8, 3 \rangle$$

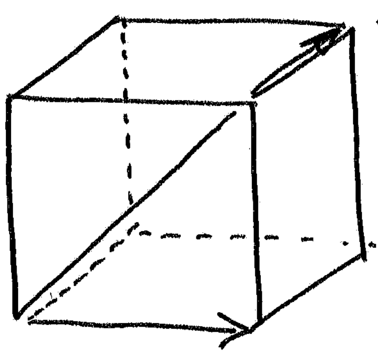
point $(1,1,1)$

normal $\langle 4, -8, 3 \rangle$

equation

$$\boxed{4(x-1) - 8(y-1) + 3(z-1) = 0}$$

(d) Find the angle between a diagonal of a cube and one of its edges.



$\vec{v} = \langle 1,1,1 \rangle$

$\vec{u} = \langle 1,0,0 \rangle$ - edge

$\vec{v} = \langle 1,1,1 \rangle$ - diagonal

$\vec{u} = \langle 1,0,0 \rangle$

$$\cos\alpha = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{1}{\sqrt{3}}$$

$$\boxed{\alpha = \arccos\left(\frac{1}{\sqrt{3}}\right)}$$

Problem 2. (10p) Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}.$$

If $(x,y)\to(0,0)$, then $t=x^2+y^2\to 0$

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = \lim_{t\to 0} \frac{e^{-t}-1}{t}$$

$$\underset{\text{l'Hospital}}{=} \lim_{t\to 0} \frac{-e^{-t}}{1} = \boxed{-1}$$

Problem 3. (10p)
(a) Find a parametrization of the curve of intersection of the surfaces $x^2 + y^2 = 1$ and $z = 4x^2$

For $x^2+y^2=1$, $\quad x=\cos t,\ y=\sin t,\ 0\leq t\leq 2\pi$

Then $\quad z=4x^2 = 4\cos^2 t$.

$$\boxed{\vec{r}(t) = \langle \cos t, \sin t, 4\cos^2 t\rangle,\quad 0\leq t\leq 2\pi}$$

(b) Find parametric equations of the tangent line to the curve in (a) at the point $\left(\sqrt{2}/2, \sqrt{2}/2, 2\right)$.

$\vec{r}(\frac{\pi}{4}) = \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4}, 4\cos^2\frac{\pi}{4}\rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\rangle$

Thus the direction of the tangent line is

$\vec{r}\,'(t) = \langle -\sin t, \cos t, -8\cos t\sin t\rangle,$

$\vec{r}\,'(\frac{\pi}{4}) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -4\rangle$

point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\right)$

direction $\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -4\rangle$

parametric equations

$$\boxed{x = \frac{\sqrt{2}}{2} - t\frac{\sqrt{2}}{2},\quad y = \frac{\sqrt{2}}{2} + t\frac{\sqrt{2}}{2},\quad z = 2-4t}$$

Problem 4. (20p) (a) Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1,1,1)$, where $x^2+y^2-2z^2+12x-8z-4=0$.

Implicit differentiation

$$2x - 4z\frac{\partial z}{\partial x} + 12 - 8\frac{\partial z}{\partial x} = 0$$

$$2x+12 = (4z+8)\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{2x+12}{4z+8} = \frac{x+6}{2z+4}$$

$$2y - 4z\frac{\partial z}{\partial y} - 8\frac{\partial z}{\partial y} = 0$$

$$2y = (4z+8)\frac{\partial z}{\partial y}, \quad \frac{\partial z}{\partial y} = \frac{2y}{4z+8} = \frac{y}{2z+4}$$

At $(1,1,1)$:

$$\boxed{\frac{\partial z}{\partial x} = \frac{7}{6}, \quad \frac{\partial z}{\partial y} = \frac{1}{6}}$$

(b) Find the equation of the tangent plane to the surface $x^2+y^2-2z^2+12x-8z-4=0$ at $(1,1,1)$.

$$F(x,y,z) = x^2+y^2-2z^2+12x-8z-4 = 0$$

$$\nabla F = \langle 2x+12,\ 2y,\ -4z-8 \rangle$$

$\nabla F(1,1,1) = \langle 14, 2, -12 \rangle$ – normal to the surface at $(1,1,1)$ so the tangent plane equation is

$$\boxed{14(x-1) + 2(y-1) - 12(z-1) = 0}$$

Problem 5. (10p) Find the curvature of $\mathbf{r}(t) = \langle \sin t, 1 - \cos t, t^2 \rangle$ at $t = 2\pi$, i.e. find $\kappa(2\pi)$.

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\vec{r}(t) = \langle \sin t, 1-\cos t, t^2 \rangle$$

$$\vec{r}\,'(t) = \langle \cos t, \sin t, 2t \rangle$$

$$\vec{r}\,''(t) = \langle -\sin t, \cos t, 2 \rangle$$

$$\vec{r}\,'(2\pi) = \langle 1, 0, 4\pi \rangle$$

$$\vec{r}\,''(2\pi) = \langle 0, 1, 2 \rangle$$

$$|\vec{r}\,'(2\pi)| = \sqrt{1+16\pi^2}$$

$$\vec{r}\,'(2\pi) \times \vec{r}\,''(2\pi) = \begin{vmatrix} i & j & k \\ 1 & 0 & 4\pi \\ 0 & 1 & 2 \end{vmatrix}$$

$$= \langle -4\pi, -2, 1 \rangle$$

$$|\vec{r}\,'(2\pi) \times \vec{r}\,''(2\pi)| = \sqrt{16\pi^2 + 4 + 1} = \sqrt{5+16\pi^2}$$

$$\kappa(2\pi) = \frac{|\vec{r}\,'(2\pi) \times \vec{r}\,''(2\pi)|}{|\vec{r}\,'(2\pi)|^3} = \boxed{\frac{\sqrt{5+16\pi^2}}{\left(\sqrt{1+16\pi^2}\right)^3}}$$

Problem 6. (10p) Find the critical points of $f(x, y) = (x^2 + y^2)e^{-x}$ and analyze them using the Second Derivative Test.

$$f = (x^2+y^2)e^{-x}$$

$$f_x = 2x e^{-x} + (x^2+y^2)e^{-x}(-1) = (2x - x^2 - y^2)e^{-x}$$

$$f_{xx} = (2-2x)e^{-x} + (2x-x^2-y^2)e^{-x}(-1) = (2-4x+x^2+y^2)e^{-x}$$

$$f_y = 2y e^{-x}$$

$$f_{xy} = f_{yx} = -2y e^{-x}, \quad f_{yy} = 2e^{-x}$$

Critical points

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \quad \begin{cases} (2x - x^2 - y^2)e^{-x} = 0 \\ 2y e^{-x} = 0 \end{cases} \quad \begin{cases} 2x - x^2 - y^2 = 0 \\ 2y = 0 \end{cases}$$

(because $e^{-x} \neq 0$)

$y = 0$, $2x - x^2 = 0$, $x(x-2) = 0$ $\quad x = 0, 2$

$(0,0)$, $(2,0)$ – critical points

$\boxed{(0,0)}$ $f_{xx} = 2 > 0$, $f_{xy} = f_{yx} = 0$, $f_{yy} = 2$

$$D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$ local minimum

$\boxed{(2,0)}$ $f_{xx} = -2e^{-2}$, $f_{xy} = f_{yx} = 0$, $f_{yy} = 2e^{-2}$

$$D = \begin{vmatrix} -2e^{-2} & 0 \\ 0 & 2e^{-2} \end{vmatrix} = -4e^{-4} < 0$$

saddle point

Problem 7. (20p) Find the absolute maximum and minimum of the function $f(x,y) = x^2 + y^2 - 8y + 3$ on the disc $x^2 + y^2 \le 9$.

$f(x,y) = x^2 + y^2 - 8y + 3$

<u>Critical points</u>

$f_x = 2x = 0, \quad f_y = 2y - 8 = 0$

$(x,y) = (0,4)$ - outside the disc.

<u>Boundary</u>

max/min $\quad f(x,y) = x^2 + y^2 - 8y + 3$

subject to $\quad g(x,y) = x^2 + y^2 = 9$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 9 \end{cases} \qquad \begin{cases} 2x = \lambda \cdot 2x \\ 2y - 8 = \lambda \cdot 2y \\ x^2 + y^2 = 9 \end{cases}$$

$2xy = \lambda \cdot 2xy$

$2yx - 8x = \lambda \cdot 2yx$

$2xy = 2yx - 8x, \quad x = 0$

$0^2 + y^2 = 9, \quad y = \pm 3$

$(0,3), (0,-3)$

$f(0,3) = -12$ - absolute minimum

$f(0,-3) = 36$ - absolute maximum.