Print your first and last name legibly above the line:

Jerry

Calculus III

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First Exam

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Problem	Possible points	Score
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
Total	100	100

Problem 1. (20p=4×5p)
(a) For what values of a are the vectors $\langle a-1, 2\rangle$ and $\langle a-4, 1\rangle$ orthogonal?

$$\langle a-1,2\rangle \cdot \langle a-4,1\rangle = (a-1)(a-4)+2$$
$$= a^2-4a-a+4+2 = a^2-5a+6$$
$$= (a-2)(a-3) = 0$$
$$\boxed{a=2 \text{ and } a=3}$$

(b) Find the angle between the planes $x+2=y-z$ and $2x-y=z$

$x-y+z+2=0$ normal $\vec{n}_1=\langle 1,-1,1\rangle$

$2x-y-z=0$ normal $\vec{n}_2=\langle 2,-1,-1\rangle$

$$\cos\alpha = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{\langle 1,-1,1\rangle\cdot\langle 2,-1,-1\rangle}{\sqrt{3}\ \sqrt{6}}$$
$$= \frac{2+1-1}{\sqrt{3}\ \sqrt{6}} = \frac{2}{3\sqrt{2}}$$
$$\boxed{\alpha = \arccos\frac{2}{3\sqrt{2}}}$$

(c) Find the equation of a plane passing through the points $A(1,1,1)$, $B(2,2,2)$, $C(1,2,3)$

$$\vec{AB} = \langle 2-1, 2-1, 2-1 \rangle = \langle 1, 1, 1 \rangle$$

$$\vec{AC} = \langle 1-1, 2-1, 3-1 \rangle = \langle 0, 1, 2 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \langle 1, -2, 1 \rangle \text{ - normal}$$

Plane passing through $C(1,1,1)$ with the normal $\langle 1, -2, 1 \rangle$

$$1 \cdot (x-1) + (-2)(y-1) + 1 \cdot (z-1) = 0$$

$$x - 1 - 2y + 2 + z - 1 = 0$$

$$\boxed{x - 2y + z = 0}$$

(d) Find the area of the triangle with vertices $A(1,1,1)$, $B(2,2,2)$, $C(1,2,3)$.

$$\text{Area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} \sqrt{1^2 + (-2)^2 + 1^2}$$

$$= \boxed{\frac{1}{2} \sqrt{6}}$$

Problem 2.

(a) Find the length of the curve $\mathbf{r}(x) = \langle x, f(x) \rangle$, $a \leq x \leq b$, where f is a given function.

$$\vec{r}\,'(x) = \langle 1, f'(x) \rangle$$

$$|\vec{r}\,'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\text{Length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

(b) Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

$$(x,y) = (t,0) \to (0,0), \quad \frac{x^2-y^2}{x^2+y^2} = \frac{t^2}{t^2} = 1 \to 1$$

$$(x,y) = (0,t) \to (0,0), \quad \frac{x^2-y^2}{x^2+y^2} = \frac{-t^2}{t^2} = -1 \to -1$$

Since we have different limits along the x-axis and along the y-axis the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

does not exist.

Problem 3. (20p=2×10p)
(a) Find the equation of the tangent plane to the surface $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ at the point $(1, 1, 2)$.

$F(x,y,z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$

$\nabla F = \langle 2x-8,\ 2y-6,\ 2z-8 \rangle$

$\nabla F(1,1,2) = \langle 2-8, 2-6, 4-8 \rangle = \langle -6,-4,-4 \rangle$

is a normal vector to the tangent plane

so the equation is

$-6(x-1) - 4(y-1) - 4(z-2) = 0$

or

$3(x-1) + 2(y-1) + 2(z-2) = 0.$

(b) Classify the surface $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ (i.e. is it ellipsoid, paraboloid, cylinder,...?)

$$(x^2 - 8x + 16) + (y^2 - 6x + 9) + (z^2 - 8x + 16) = 17$$

$$(x-4)^2 + (y-3)^2 + (z-4)^2 = 17$$

Sphere of radius $\sqrt{17}$ centered at $(4, 3, 4)$.

Problem 4. (20p) Find the maximum and minimum values of $f(x, y) = (x-1)^2 + (y-2)^2$ on the disc $x^2 + y^2 \leq 45$.

Critical points

$$\begin{cases} f_x = 2(x-1) = 0 \\ f_y = 2(y-2) = 0 \end{cases}$$

$(x, y) = (1, 2)$ is inside the disc $x^2 + y^2 \leq 45$

$$f(1, 2) = \boxed{0}$$

On the boundary

$$\begin{cases} \text{max/min } f(x, y) = (x-1)^2 + (y-2)^2 \\ \text{subject to } g(x, y) = x^2 + y^2 = 45 \end{cases}$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 45 \end{cases} \qquad \begin{cases} 2(x-1) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ x^2 + y^2 = 45 \end{cases}$$

$$(x-1)y = \lambda xy = (y-2)x$$

$$\cancel{xy} - y = \cancel{yx} - 2x$$

$$y = 2x$$

$x^2 + (2x)^2 = 45$, $5x^2 = 45$, $x^2 = 9$, $x = \pm 3$, $y = \pm 6$

$(3, 6)$ & $(-3, -6)$

$$f(3, 6) = 2^2 + 4^2 = \boxed{20}$$

$$f(-3, -6) = (-4)^2 + (-8)^2 = 16 + 64 = \boxed{80}$$

$$\boxed{\begin{array}{lll} \text{Absolute minimum} & f(1,2) = 0 \\ \text{Absolute maximum} & f(-3,-6) = 80 \end{array}}$$

Problem 5. (20p=2×10p) Using the method of Lagrange multipliers find the distance of the point $(17, -4, -3)$ to the plane $6x - 3y + 2z = 10$.

Equivalent problem: find the minimum of the square of the distance

$$f(x,y,z) = (x-17)^2 + (y+4)^2 + (z+3)^2$$

between $(17,-4,-3)$ and a point on the plane $6x-3y+2z=10$.

Thus we have the problem

$$\begin{cases} \text{minimum of } f(x,y,z) = (x-17)^2 + (y+4)^2 + (z+3)^2 \\ \text{subject to } \quad g(x,y,z) = 6x - 3y + 2z = 10 \end{cases}$$

From the geometric considerations we know that the minimum is attained and by the Lagrange multiplier theorem it is attained at a point where

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 10 \end{cases} \quad \text{i.e.} \quad \begin{cases} 2(x-17) = \lambda \cdot 6 \\ 2(y+4) = \lambda \cdot (-3) \\ 2(z+3) = \lambda \cdot 2 \\ 6x - 3y + 2z = 10 \end{cases}$$

Solving the first three equations gives

$$(*) \begin{cases} x = 3\lambda + 17 \\ y = -\frac{3}{2}\lambda - 4 \\ z = \lambda - 3 \end{cases}$$

so the last equation gives

$$6(3\lambda+17) - 3\left(-\tfrac{3}{2}\lambda - 4\right) + 2(\lambda - 3) = 10$$

Hence $\lambda = -4$

Now $(*)$ yields $x=5$, $y=2$, $z=-7$

Thus the minimal distance is

$$d = \sqrt{f(5,2,-7)} = \sqrt{196} = 14$$

The point on the plane that is closest to $(17,-4,-3)$ is $(5,2,-7)$.