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Calculus III

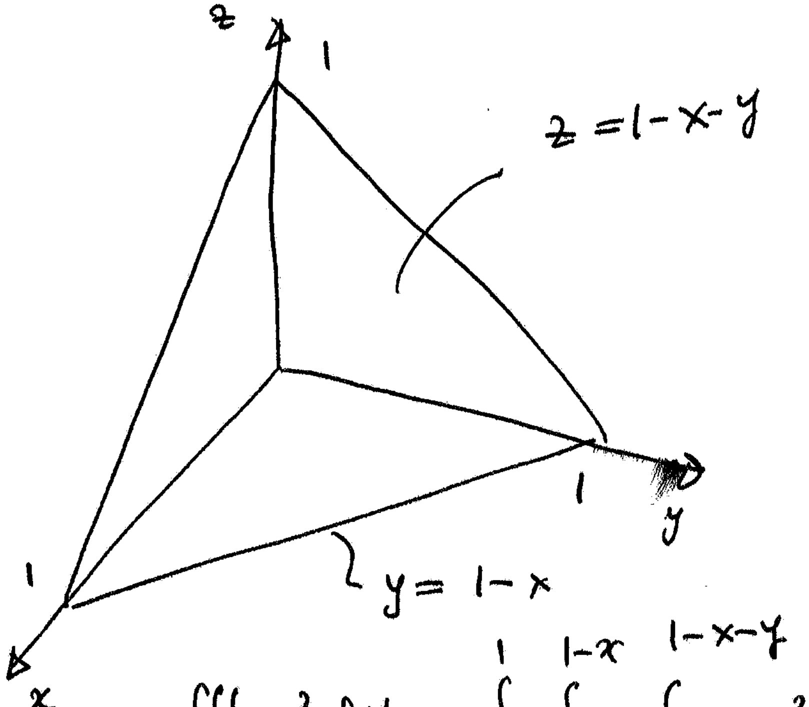
Professor Piotr Hajłasz

Second Second Exam

November 16, 2015.

Problem	Possible points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1. Evaluate the integral $\iiint_T x^2\,dV$, where T is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.



$$\iiint_T x^2\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} x^2 z\Big|_{z=0}^{z=1-x-y} dy\,dx = \int_0^1\int_0^{1-x} x^2(1-x-y)\,dy\,dz = \int_0^1\int_0^{1-x} x^2-x^3-x^2y\,dy\,dx$$

$$= \int_0^1 \left(x^2y - x^3y - \frac{x^2y^2}{2}\right)\Big|_{y=0}^{y=1-x} dx = \int_0^1 x^2(1-x) - x^3(1-x) - \frac{x^2(1-x)^2}{2}\,dx$$

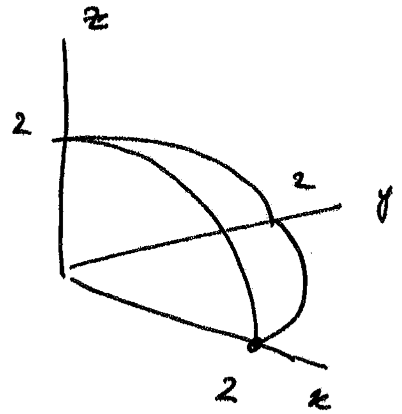
$$= \int_0^1 x^2 - x^3 - x^3 + x^4 - \frac{x^2(1-2x+x^2)}{2}\,dx = \int_0^1 x^2-2x^3+x^4-\frac{x^2}{2}+x^3-\frac{x^4}{2}\,dx$$

$$= \int_0^1 \frac{x^2}{2} - x^3 + \frac{x^4}{2}\,dx = \frac{x^3}{6}-\frac{x^4}{4}+\frac{x^5}{10}\Big|_0^1$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{10}{60}-\frac{15}{60}+\frac{6}{60} = \boxed{\frac{1}{60}}$$

Problem 2. Evaluate the integral

$$I = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$



We integrate ~~of~~ over the part of the ball of radius 2 that is contained in the first octant. In spherical coordinates it is

$$D = \left\{ (\rho, \theta, \phi) \,\middle|\, 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \rho \le 2 \right\}$$

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2} \underbrace{\rho^2 \cos^2\phi}_{z^2} \cdot \underbrace{\rho}_{\sqrt{x^2+y^2+z^2}} \cdot \underbrace{\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}_{dV}$$

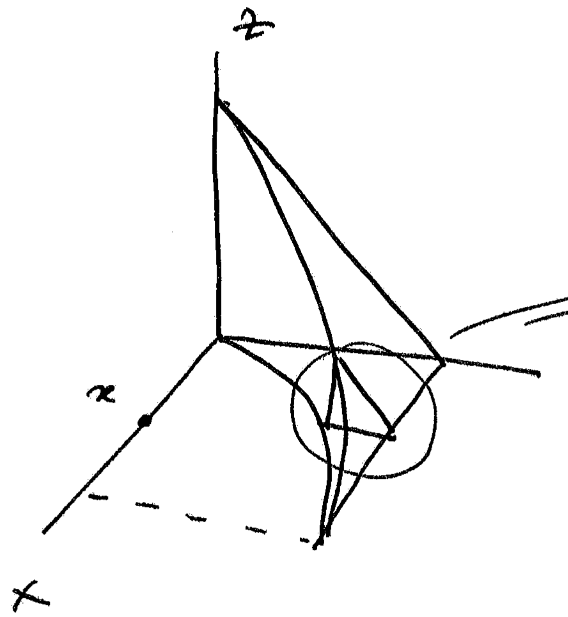
$$= \int_0^{\pi/2} d\theta \cdot \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi \cdot \int_0^{2} \rho^5 \, d\rho =$$

$$\frac{\pi}{2} \cdot \left[\frac{\cos^3\phi}{-3} \right]_0^{\pi/2} \quad \frac{\rho^6}{6}\Big|_0^2 =$$

$$= \frac{\pi}{2} \cdot \frac{1}{3} \cdot \frac{64}{6} = \frac{64\pi}{4 \cdot 9} = \boxed{\frac{16\pi}{9}}$$

Problem 3. Rewrite the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx \quad \text{as} \quad \int_?^? \int_?^? \int_?^? f(x,y,z)\, dy\, dz\, dx.$$

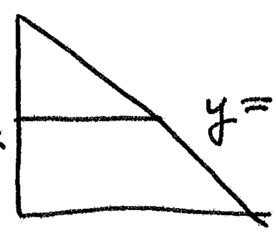


$$\boxed{0 \leq x \leq 1}$$

$$z = 1-y = 1-\sqrt{x}$$

$y = \sqrt{x}$

$$\boxed{0 \leq z \leq 1-\sqrt{x}}$$



$$\boxed{\sqrt{x} \leq y \leq 1-z}$$

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

Problem 4. Find a function f such that $\mathbf{F} = \nabla f$ and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = yze^{xz}\,\mathbf{i} + (e^{xz} + e^{y})\,\mathbf{j} + (xye^{xz} + 2z)\,\mathbf{k}$ and C is parametrized by $\mathbf{r}(t) = (t^2+1)\,\mathbf{i} + (t^2-1)\,\mathbf{j} + (t^2-2t)\,\mathbf{k}$, $0 \le t \le 2$.

$$\begin{cases} f_x = yz\,e^{xz} \\ f_y = e^{xz} + e^{y} \\ f_z = xy\,e^{xz} + 2z \end{cases}$$

$$f = \int yz\,e^{xz}\,dx = y\,e^{xz} + g(y,z)$$

$$f_y = e^{xz} + g_y(y,z) = e^{xz} + e^{y}$$

$$g_y(y,z) = e^{y}$$

$$g(y,z) = e^{y} + h(z)$$

$$f = y e^{xz} + e^{y} + h(z)$$

$$f_z = xy e^{xz} + h'(z) = xy\,e^{xz} + 2z$$

$$h'(z) = 2z$$

$$h(z) = z^2 \quad (+k)$$

Take $k = 0$.

$$f = y e^{xz} + e^{y} + z^2$$

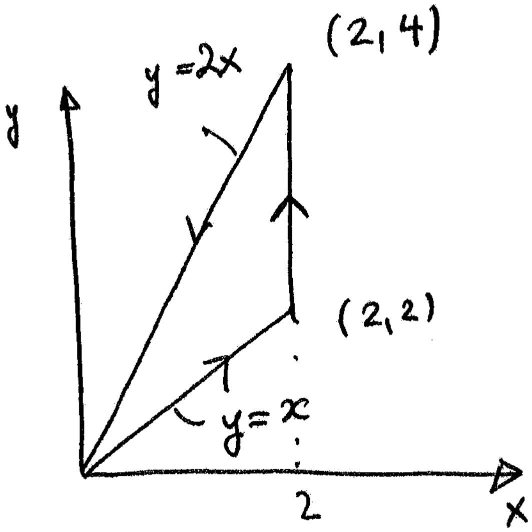
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(0)) = \heartsuit$$

$$\vec{r}(2) = \langle 5, 3, 0 \rangle, \quad \vec{r}(0) = \langle 1, -1, 0 \rangle$$

$$\heartsuit = f(5,3,0) - f(1,-1,0) = 3e^{5\cdot 0} + e^{3} + 0^2 - (-e^{1\cdot 0} + e^{-1} + 0^2)$$

$$= 3 + e^{3} + 1 - e^{-1} = \boxed{4 + e^{3} - \frac{1}{e}}$$

Problem 5. Use Green's theorem to evaluate $\int_C xy^2\,dx + 2x^2y\,dy$, where C is the positively oriented triangle with vertices $(0,0)$, $(2,2)$ and $(2,4)$.



$$\int_C \underbrace{xy^2}_{P}\,dx + \underbrace{2x^2y}_{Q}\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D (4xy - 2xy)\,dA = \int_0^2 \int_x^{2x} 2xy\,dy\,dx$$

$$= \int_0^2 xy^2 \Big|_{y=x}^{y=2x} dx = \int_0^2 x(2x)^2 - x \cdot x^2\,dx$$

$$= \int_0^2 3x^3\,dx = \frac{3x^4}{4}\Big|_0^2 = \boxed{12}$$